Linear Algebra and its Applications 433 (2010) 263–296

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Equalities and inequalities for inertias of Hermitian matrices with applications

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ARTICLE INFO

Article history:
Received 9 September 2009
Accepted 18 February 2010
Available online 27 March 2010

Submitted by R. Loewy

AMS classification:
15A09
15A24
15A45
15A57

Keywords:
Hermitian matrix
Block matrix
Matrix expression
Matrix equation
Hermitian solution
Rank
Inertia
Moore–Penrose inverse
Equality
Inequality

ABSTRACT

The inertia of a Hermitian matrix is defined to be a triplet composed of the numbers of the positive, negative and zero eigenvalues of the matrix counted with multiplicities, respectively. In this paper, we show some basic formulas for inertias of 2×2 block Hermitian matrices. From these formulas, we derive various equalities and inequalities for inertias of sums, parallel sums, products of Hermitian matrices, submatrices in block Hermitian matrices, differences of outer inverses of Hermitian matrices. As applications, we derive the extremal inertias of the linear matrix expression $A - BXB^*$ with respect to a variable Hermitian matrix X. In addition, we give some results on the extremal inertias of Hermitian solutions to the matrix equation $AX = B$, as well as the extremal inertias of a partial block Hermitian matrix.

1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ and $\mathbb{C}_h^{m\times m}$ stand for the sets of all $m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices, respectively; the symbols A^T, A^*, $r(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ stand for

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2010.02.018

the transpose, conjugate transpose, rank, range (column space) and null space of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; I_m denotes the identity matrix of order m. We write $A > 0$ (or $A \geqslant 0$) if A is Hermitian positive (or nonnegative) definite. Two Hermitian matrices A and B of the same size are said to satisfy the inequality $A > B$ (or $A \geqslant B$) in the Löwner partial ordering if $A - B$ is positive (or nonnegative) definite. The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is defined to be the unique solution X to the four matrix equations

$$\text{(i) } AXA = A, \quad \text{(ii) } XAX = X, \quad \text{(iii) } (AX)^* = AX, \quad \text{(iv) } (XA)^* = XA.$$

A matrix X is called an outer inverse of A if it satisfies (ii). Further, the symbols E_A and F_A stand for the two orthogonal projectors $E_A = I_m - AA^{\dagger}$ and $F_A = I_n - A^{\dagger}A$ onto the null spaces A^* and A, respectively. A well-known property of the Moore–Penrose inverse is $(A^{\dagger})^* = (A^*)^{\dagger}$. In addition, $AA^{\dagger} = A^{\dagger}A$ if $A = A^*$. We shall repeatedly use them in the latter part of this paper. Results on the Moore–Penrose inverse can be found, e.g., in [4,5,23].

As is well known, the eigenvalues of a Hermitian matrix $A \in \mathbb{C}_h^{m\times m}$ are all real, and the inertia of A is defined to be the triplet

$$\text{In}(A) = \{i_+(A), i_-(A), i_0(A)\},$$

where $i_+(A)$, $i_-(A)$ and $i_0(A)$ are the numbers of the positive, negative and zero eigenvalues of A counted with multiplicities, respectively. The two numbers $i_+(A)$ and $i_-(A)$ are usually called the positive index and the negative index of inertia, or the positive and negative signatures, respectively. The difference $i_+(A) - i_-(A)$, denoted by $s(A)$, is called the signature of A. Both $i_+(A)$ and $i_-(A)$ can easily be computed by elementary congruence matrix operations. In linear algebra, one of the most fundamental quantities is the rank of a matrix, which is a well understood number and can easily be computed by elementary matrix operations. For a Hermitian matrix A, we have

$$r(A) = i_+(A) + i_-(A). \tag{1.1}$$

In the investigation of ranks of matrices, a seminal paper was given by Marsaglia and Styan [29], in which, many equalities and inequalities for ranks of matrices and their applications were given. This paper brought some essential influence to the development of matrix theory, and a huge amount of results in matrix theory and applications were derived from the equalities and inequalities in the paper. A group of well-known formulas for ranks of block matrices from [29] are given in the following lemma.

Lemma 1.1. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$ be given. Then*

$$r[A, B] = r(A) + r(E_AB) = r(B) + r(E_BA), \tag{1.2}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{1.3}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_BAF_C), \tag{1.4}$$

$$r\begin{bmatrix} \pm AA^* & B \\ B^* & 0 \end{bmatrix} = r[A, B] + r(B), \tag{1.5}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & E_AB \\ CF_A & D - CA^{\dagger}B \end{bmatrix}, \tag{1.6}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r[A, B] - r(A) + r[E_{C_1}(D - CA^{\dagger}B)F_{B_1}], \tag{1.7}$$

where $B_1 = E_AB$ and $C_1 = CF_A$. If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(A^*)$, then

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(D - CA^{\dagger}B). \tag{1.8}$$

The following result is derived from (1.8) and the well-known formula $A^{\dagger} = A^*(A^*AA^*)^{\dagger}A^*$ in [39].

Lemma 1.2 [36]. *Let* $A \in \mathbb{C}^{m\times n},\ B \in \mathbb{C}^{m\times k},\ C \in \mathbb{C}^{l\times n}$ *and* $D \in \mathbb{C}^{l\times k}$ *be given. Then*

$$r(D - CA^{\dagger}B) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A). \tag{1.9}$$

Note that the inertia of a Hermitian matrix divides the eigenvalues of the matrix into three parts on the real line. Hence the inertia of a Hermitian matrix can be used to characterize definiteness of the Hermitian matrix. The following results are obvious from the definitions of the rank and inertia of a matrix.

Lemma 1.3. *Let* $A \in \mathbb{C}^{m\times m},\ B \in \mathbb{C}^{m\times n}$ *and* $C \in \mathbb{C}_h^{m\times m}$*. Then,*

(a) *A is nonsingular if and only if* $r(A) = m$.
(b) $B = 0$ *if and only if* $r(B) = 0$.
(c) $C > 0$ $(C < 0)$ *if and only if* $i_+(C) = m$ $(i_-(C) = m)$.
(d) $C \geqslant 0$ $(C \leqslant 0)$ *if and only if* $i_-(C) = 0$ $(i_+(C) = 0)$.

Lemma 1.4. *Let* $\mathcal{S}$ *be a set consisting of (square) matrices over* $\mathbb{C}^{m\times n}$*, and let* $\mathcal{H}$ *be a set consisting of Hermitian matrices over* $\mathbb{C}_h^{m\times m}$*. Then,*

(a) $\mathcal{S}$ *has a nonsingular matrix if and only if* $\max_{X\in\mathcal{S}} r(X) = m$.
(b) *Any* $X \in \mathcal{S}$ *is nonsingular if and only if* $\min_{X\in\mathcal{S}} r(X) = m$.
(c) $0 \in \mathcal{S}$ *if and only if* $\min_{X\in\mathcal{S}} r(X) = 0$.
(d) $\mathcal{S} = \{0\}$ *if and only if* $\max_{X\in\mathcal{S}} r(X) = 0$.
(e) $\mathcal{H}$ *has a matrix* $X > 0$ $(X < 0)$ *if and only if* $\max_{X\in\mathcal{H}} i_+(X) = m(\max_{X\in\mathcal{H}} i_-(X) = m)$.
(f) *Any* $X \in \mathcal{H}$ *satisfies* $X > 0$ $(X < 0)$ *if and only if* $\min_{X\in\mathcal{H}} i_+(X) = m(\min_{X\in\mathcal{H}} i_-(X) = m)$.
(g) $\mathcal{H}$ *has a matrix* $X \geqslant 0$ $(X \leqslant 0)$ *if and only if* $\min_{X\in\mathcal{H}} i_-(X) = 0(\min_{X\in\mathcal{H}} i_+(X) = 0)$.
(h) *Any* $X \in \mathcal{H}$ *satisfies* $X \geqslant 0$ $(X \leqslant 0)$ *if and only if* $\max_{X\in\mathcal{H}} i_-(X) = 0$ $(\max_{X\in\mathcal{H}} i_+(X) = 0)$.

These two lemmas show that if some formulas for the (extremal) rank and the positive and negative signatures of a Hermitian matrix are derived, then we can use them as tools to characterize equalities and inequalities for the Hermitian matrix. This basic algebraic method, referred to as the matrix rank/inertia method, is available for studying various Hermitian matrix expressions that involve generalized inverses of matrices and variable matrices.

In order to derive some explicit formulas for inertias of Hermitian matrices, we need to use the following three types of elementary block congruence matrix operation (EBCMO, for short) for a block Hermitian matrix with the same row and column partition:

(I) interchange ith and jth block rows, while interchange ith and jth block columns in the block Hermitian matrix;
(II) multiply ith block row by a nonsingular matrix P from the left-hand side, while multiply ith block column by P^* from the right-hand side in the block Hermitian matrix;
(III) add ith block row multiplied by a matrix P from the left-hand side to jth block row, while add ith block column multiplied by P^* from the right-hand side to jth block column in the block Hermitian matrix.

The three types of operation are in fact equivalent to some congruence transformation of a Hermitian matrix $A \to PAP^*$, where the nonsingular matrix P is from the elementary block matrix operations to the block rows of A, and P^* is from the elementary block matrix operations to the block columns of A. Some examples of the matrix P corresponding to the EBCMOs can be found in the proofs of Lemmas 1.6, 2.1, 2.2 and below Theorem 2.3. In fact, the congruence operations produced from the EBCMOs were widely used by some authors in the investigations of inertias of block Hermitian matrices; see, e.g., [6,8,9,12,13,21,22,30]. Because the matrix P is nonsingular, the equality $\mathrm{In}(A) = \mathrm{In}(PAP^*)$ holds by the well-known Sylvester's law of inertia in (1.10). Since the EBCMOs do not change the inertia of a Hermitian matrix, we shall repeatedly use the EBCMOs to simplify block Hermitian matrices and to establish equalities for their inertias in the latter part of this paper. We also use the following simple or known results on positive and negative signatures of Hermitian matrices.

Lemma 1.5. *Let $A \in \mathbb{C}_h^{m\times m}$, $B \in \mathbb{C}_h^{n\times n}$, $Q \in \mathbb{C}^{m\times n}$, and assume that $P \in \mathbb{C}^{m\times m}$ is nonsingular. Then*

$$i_{\pm}(PAP^*) = i_{\pm}(A), \tag{1.10}$$

$$i_{\pm}(A^{\dagger}) = i_{\pm}(A) \quad \text{and} \quad i_{\pm}(A^{2k+1}) = i_{\pm}(A), \tag{1.11}$$

$$i_{\pm}(\lambda A) = \begin{cases} i_{\pm}(A) & \text{if } \lambda > 0, \\ i_{\mp}(A) & \text{if } \lambda < 0, \end{cases} \tag{1.12}$$

$$i_{\pm}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = i_{\pm}(A) + i_{\pm}(B), \tag{1.13}$$

$$i_{\pm}\begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = r(Q), \tag{1.14}$$

$$i_{\pm}\begin{bmatrix} A & Q \\ Q^* & B \end{bmatrix} \geqslant \max\{i_{\pm}(A), i_{\pm}(B)\}. \tag{1.15}$$

Eq. (1.10) is the well-known Sylvester's law of inertia. Note that the signs of nonzero eigenvalues of A, $A^{\dagger}$ and A^{2k+1} are the same. Hence (1.11) holds. Eqs. (1.12) and (1.13) are obvious from the definition of inertia. Eq. (1.15) is the well-known Poincare's inequality (see, e.g., [14,21]), and (1.14) is well known (see, e.g., [21,22]).

Lemma 1.6. *Let $A \in \mathbb{C}_h^{m\times m}$, $B \in \mathbb{C}_h^{n\times n}$, and $P, Q \in \mathbb{C}^{m\times n}$. Then*

$$i_{\pm}(P^*AP) \leqslant i_{\pm}(A). \tag{1.16}$$

In particular,

(a) *$r(P^*AP) = r(A)$ if and only if $i_+(P^*AP) = i_+(A)$ and $i_-(P^*AP) = i_-(A)$.*
(b) *If $P^*AP = B$ and $QBQ^* = A$, then $i_{\pm}(A) = i_{\pm}(B)$ and $r(A) = r(B)$.*

Proof. It follows from (1.15) that

$$i_{\pm}\begin{bmatrix} A & AP \\ P^*A & P^*AP \end{bmatrix} \geqslant i_{\pm}(P^*AP). \tag{1.17}$$

It is also easy to verify that

$$\begin{bmatrix} I_m & 0 \\ -P^* & I_n \end{bmatrix}\begin{bmatrix} A & AP \\ P^*A & P^*AP \end{bmatrix}\begin{bmatrix} I_m & -P \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence we have

$$i_{\pm}\begin{bmatrix} A & AP \\ P^*A & P^*AP \end{bmatrix} = i_{\pm}(A) \tag{1.18}$$

by (1.10). Combining (1.17) and (1.18) leads to (1.16). Also note that

$$r(A) - r(P^*AP) = [i_+(A) - i_+(P^*AP)] + [i_-(A) - i_-(P^*AP)].$$

Setting both sides of the equality to zero leads to the result in (a). Applying (1.16) to $P^*AP = B$ and $QBQ^* = A$ yields $i_{\pm}(A) \geqslant i_{\pm}(B)$ and $i_{\pm}(A) \leqslant i_{\pm}(B)$. Hence $i_{\pm}(A) = i_{\pm}(B)$. $\square$

For $A, B \in \mathbb{C}_h^{m\times m}$, the sum $A + B$ can be written as

$$A + B = [I_m, I_m]\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} I_m \\ I_m \end{bmatrix}.$$

Applying Lemma 1.6 and (1.13) to the product on the right-hand side of the above equality leads to the following known result.

Lemma 1.7 [17,31]. *Let $A, B \in \mathbb{C}_h^{m\times m}$. Then*

$$i_{\pm}(A+B) \leqslant i_{\pm}(A) + i_{\pm}(B). \tag{1.19}$$

In particular, the following statements are equivalent:

(a) $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$.
(b) $r(A+B) = r(A) + r(B)$.
(c) $i_+(A+B) = i_+(A) + i_+(B)$ *and* $i_-(A+B) = i_-(A) + i_-(B)$.

Note that $A = (A+B) + (-B)$. Hence we see from (1.19) that if $B \geqslant 0$, then

$$i_+(A) \leqslant i_+(A+B) \quad \text{and} \quad i_-(A) \geqslant i_-(A+B).$$

These two inequalities were given in Lemma 2 of [30].

In addition, we use the following simple or well-known results to simplify various rank equalities for matrices:

$$\mathcal{R}(B) \subseteq \mathcal{R}(A) \Leftrightarrow r[A, B] = r(A), \tag{1.20}$$

$$\mathcal{R}(A) \subseteq \mathcal{R}(B) \quad \text{and} \quad r(A) = r(B) \Rightarrow \mathcal{R}(A) = \mathcal{R}(B), \tag{1.21}$$

$$\mathcal{R}(A) \subseteq \mathcal{R}(B) \Rightarrow \mathcal{R}(PA) \subseteq \mathcal{R}(PB), \tag{1.22}$$

$$\mathcal{R}(A) = \mathcal{R}(AA^*) = \mathcal{R}(AA^*A) = \mathcal{R}(AA^{\dagger}) = \mathcal{R}[(A^{\dagger})^*], \tag{1.23}$$

$$\mathcal{R}(A^*) = \mathcal{R}(A^*A) = \mathcal{R}(A^*AA^*) = \mathcal{R}(A^{\dagger}) = \mathcal{R}(A^{\dagger}A), \tag{1.24}$$

$$\mathcal{R}(AB^{\dagger}B) = \mathcal{R}(AB^{\dagger}) = \mathcal{R}(AB^*B) = \mathcal{R}(AB^*), \tag{1.25}$$

$$\mathcal{R}(A_1) = \mathcal{R}(A_2) \text{ and } \mathcal{R}(B_1) = \mathcal{R}(B_2) \Rightarrow r[A_1, B_1] = r[A_2, B_2]. \tag{1.26}$$

The paper is organized as follows. In Section 2, we derive some equalities and inequalities for positive and negative signatures of 2×2 block Hermitian matrices, and give various direct consequences of these formulas. In Section 3, we give two formulas for the positive and negative signatures of a Hermitian Schur complement, and use them to derive various equalities and inequalities for the rank and the positive and negative signatures of sums, differences, and products of Hermitian matrices and their Moore–Penrose inverses. In Section 4, we give the maximal and minimal values of the rank and the positive and negative signatures of the matrix expression $A - BXB^*$ with respect to a variable Hermitian matrix X. Section 5 gives the maximal and minimal values of the rank and the positive and negative signatures of the Hermitian solution to the matrix equation $AX = B$. Section 6 gives the maximal and minimal values of the rank and the positive and negative signatures of a given partial Hermitian matrix. Section 7 derives the maximal and minimal values of the positive and negative signatures of $A - B_1X_1B_1^* - \cdots - B_kX_kB_k^*$ with respect to variable Hermitian matrices $X_1, \ldots, X_k$. In Section 8, we propose some problems on inertias of Hermitian matrices for further consideration.

2. Equalities and inequalities for inertias of block Hermitian matrices

We first show some simple formulas for positive and negative signatures of block Hermitian matrices by using the $*$-congruence transformation.

Lemma 2.1. *Let $A, B \in \mathbb{C}_h^{m\times m}$. Then*

$$i_{\pm}\begin{bmatrix} A & B \\ B & A \end{bmatrix} = i_{\pm}(A+B) + i_{\pm}(A-B). \tag{2.1}$$

Proof. It is easy to verify that

$$\frac{1}{2}\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}\begin{bmatrix} A & B \\ B & A \end{bmatrix}\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix} = \begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}. \tag{2.2}$$

Applying (1.10) and (1.13) to (2.2) leads to (2.1). □

Lemma 2.2. *Let $A \in \mathbb{C}_h^{m\times m}$, $B \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}_h^{n\times n}$ be given. Then*

$$i_{\pm}\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} = i_{\pm}\begin{bmatrix} A & -B \\ -B^* & D \end{bmatrix} = i_{\mp}\begin{bmatrix} -A & B \\ B^* & -D \end{bmatrix}. \tag{2.3}$$

Proof. Applying (1.10) and (1.12) to the following two equalities

$$\begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix}\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}\begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix} = \begin{bmatrix} A & -B \\ -B^* & D \end{bmatrix} \quad \textit{and} \quad -\begin{bmatrix} A & -B \\ -B^* & D \end{bmatrix} = \begin{bmatrix} -A & B \\ B^* & -D \end{bmatrix}$$

yields (2.3). □

The main result of this section is given below.

Theorem 2.3. *Let $A \in \mathbb{C}_h^{m\times m}$, $B \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}_h^{n\times n}$, and denote*

$$M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad N = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}, \quad S_A = D - B^*A^{\dagger}B, \quad B_1 = E_AB. \tag{2.4}$$

Then,

(a) *The following equalities hold*

$$i_{\pm}(M) = r(B) + i_{\pm}(E_BAE_B), \tag{2.5}$$

$$i_{\pm}(M) = i_{\pm}(A) + i_{\pm}\begin{bmatrix} 0 & E_AB \\ B^*E_A & -B^*A^{\dagger}B \end{bmatrix}, \tag{2.6}$$

$$i_{\pm}(M) = r[A, B] - i_{\mp}(A) + i_{\mp}(F_{B_1}B^*A^{\dagger}BF_{B_1}), \tag{2.7}$$

$$i_{\pm}(N) = i_{\pm}(A) + i_{\pm}\begin{bmatrix} 0 & E_AB \\ B^*E_A & S_A \end{bmatrix}, \tag{2.8}$$

$$i_{\pm}(N) = r[A, B] - i_{\mp}(A) + i_{\pm}(F_{B_1}S_AF_{B_1}). \tag{2.9}$$

(b) *The following equalities hold*

$$i_{\pm}(M) = r(M) - r(B) - i_{\mp}(E_BAE_B), \tag{2.10}$$

$$i_{\pm}(M) = r(M) - i_{\mp}(A) - i_{\mp}\begin{bmatrix} 0 & E_AB \\ B^*E_A & -B^*A^{\dagger}B \end{bmatrix}, \tag{2.11}$$

$$i_{\pm}(M) = r(M) - r[A, B] + i_{\pm}(A) - i_{\pm}(F_{B_1}B^*A^{\dagger}BF_{B_1}), \tag{2.12}$$

$$i_{\pm}(N) = r(N) - i_{\mp}(A) - i_{\mp}\begin{bmatrix} 0 & E_AB \\ B^*E_A & S_A \end{bmatrix}, \tag{2.13}$$

$$i_{\pm}(N) = r(N) - r[A, B] + i_{\pm}(A) - i_{\pm}(F_{B_1}S_AF_{B_1}). \tag{2.14}$$

(c) *The following inequalities hold*

$$i_{\pm}(M) \leqslant \min\{r(M) - r(B),\ r(M) - r[A, B] + i_{\pm}(A),\ r[A, B] - i_{\mp}(A) + i_{\mp}(B^*A^{\dagger}B)\}, \tag{2.15}$$

$$i_{\pm}(M) \geqslant \max\{r(B), r[A, B] - i_{\mp}(A),\ i_{\pm}(A) + i_{\mp}(B^*A^{\dagger}B)\}, \tag{2.16}$$

$$i_{\pm}(N) \leqslant \min\{r(N) - r[A, B] + i_{\pm}(A), r[A, B] - i_{\mp}(A) + i_{\pm}(S_A)\}, \tag{2.17}$$
$$i_{\pm}(N) \geqslant \max\{r[A, B] - i_{\mp}(A), r(N) - r[A, B] + i_{\pm}(A) - i_{\pm}(S_A)\}. \tag{2.18}$$

In particular,

(d) *If* $A \geqslant 0$, *then* $i_+(M) = r[A, B]$ *and* $i_-(M) = r(B)$.
(e) *If* $A \leqslant 0$, *then* $i_+(M) = r(B)$ *and* $i_-(M) = r[A, B]$.
(f) *If* $A > 0$, *then* $i_+(M) = m$ *and* $i_-(M) = r(B)$.
(g) *If* $A < 0$, *then* $i_+(M) = r(B)$ *and* $i_-(M) = m$.
(h) $i_{\pm}(M) = m$ *if and only if* $i_{\mp}(E_BAE_B) = 0$ *and* $r(E_BAE_B) = r(E_B)$.
(i) $i_{\pm}(M) = r(B)$ *if and only if* $i_{\pm}(E_BAE_B) = 0$.
(j) $i_{\pm}(M) = 0$ *if and only if* $i_{\pm}(A) = 0$ *and* $B = 0$.
(k) *If* $\mathcal{R}(A) \subseteq \mathcal{R}(B)$, *then* $i_{\pm}(M) = r(B)$.
(l) *If* $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, *then* $i_{\pm}(M) = i_{\pm}(A) + r(B)$.
(m) *If* $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, *then* $i_{\pm}(N) = i_{\pm}(A) + i_{\pm}(D - B^*A^{\dagger}B)$.
(n) *If* $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ *and* $\mathcal{R}(B^*) \cap \mathcal{R}(D) = \{0\}$, *then* $i_{\pm}(N) = i_{\pm}(A) + i_{\pm}(D) + i_{\mp}(B^*A^{\dagger}B)$.
(o) *If* $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ *and* $\mathcal{R}(B^*) \cap \mathcal{R}(D) = \{0\}$, *then* $i_{\pm}(N) = i_{\pm}(A) + r(B) + i_{\pm}(D)$.
(p) $i_{\pm}(N) = i_{\pm}(A)$ *if and only if* $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ *and* $i_{\pm}(D - B^*A^{\dagger}B) = 0$.
(q) $i_{\pm}(N) = 0 \Leftrightarrow i_{\pm}(A) = 0, \mathcal{R}(B) \subseteq \mathcal{R}(A)$ *and* $i_{\pm}(D - B^*A^{\dagger}B) = 0 \Leftrightarrow i_{\pm}(D) = 0, \mathcal{R}(B^*) \subseteq \mathcal{R}(D)$ *and* $i_{\pm}(A - BD^{\dagger}B^*) = 0$.

Proof. Let $P = \begin{bmatrix} I_m & -\left(I_m - \frac{1}{2}BB^{\dagger}\right)A(B^{\dagger})^* \\ 0 & I_n \end{bmatrix}$. Then it is easy to verify by (1.4) that

$$P\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}P^* = \begin{bmatrix} E_BAE_B & B \\ B^* & 0 \end{bmatrix} = \begin{bmatrix} E_BAE_B & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix},$$
$$r\begin{bmatrix} E_BAE_B & B \\ B^* & 0 \end{bmatrix} = r\begin{bmatrix} E_BAE_B & 0 \\ 0 & 0 \end{bmatrix} + r\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}.$$

Applying (1.13), (1.14) and Lemma 1.7(b) and (c) gives

$$i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} = i_{\pm}\begin{bmatrix} E_BAE_B & B \\ B^* & 0 \end{bmatrix} = i_{\pm}\begin{bmatrix} E_BAE_B & 0 \\ 0 & 0 \end{bmatrix} + i_{\pm}\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} = r(B) + i_{\pm}(E_BAE_B),$$

establishing (2.5). It is also easy to verify by (1.6) that

$$\begin{bmatrix} I_m & 0 \\ -B^*A^{\dagger} & I_n \end{bmatrix}\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}\begin{bmatrix} I_m & -A^{\dagger}B \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} A & E_AB \\ B^*E_A & S_A \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & E_AB \\ B^*E_A & S_A \end{bmatrix},$$
$$r\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} = r\begin{bmatrix} A & E_AB \\ B^*E_A & S_A \end{bmatrix} = r\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + r\begin{bmatrix} 0 & E_AB \\ B^*E_A & S_A \end{bmatrix}.$$

Eq. (2.8) follows from these two equalities, Lemma 1.7(b) and (c), and (1.13). Applying (2.5) to the block matrix on the right-hand side of (2.8) and simplifying by (1.2) gives

$$\begin{aligned} i_{\pm}(N) &= i_{\pm}(A) + r(E_AB) + i_{\pm}(F_{B_1}S_AF_{B_1}) \\ &= i_{\pm}(A) + r[A, B] - r(A) + i_{\pm}(F_{B_1}S_AF_{B_1}) \\ &= r[A, B] - i_{\mp}(A) + i_{\pm}(F_{B_1}S_AF_{B_1}), \end{aligned}$$

establishing (2.9). Setting $D = 0$ in (2.8) and (2.9) leads to (2.6) and (2.7). Adding $i_{\mp}(M)$ and $i_{\mp}(N)$ on the both sides of (2.5)–(2.9) respectively leads to (2.10)–(2.14). The inequalities in (2.15)–(2.18) follow from (2.5)–(2.14).

If $A \geqslant 0$, then $E_BAE_B \geqslant 0$. Hence $i_+(E_BAE_B) = r(E_BAE_B) = r(E_BA) = r[A, B] - r(B)$ by (1.2) and $i_-(E_BAE_B) = 0$. Substituting them into (2.5) yields (d). Result (e) can be shown similarly.

If $A > 0$, then $E_BAE_B \geqslant 0$. Hence $i_+(E_BAE_B) = r(E_BAE_B) = r(E_B) = m - r(B)$ by (1.2) and $i_-(E_BAE_B) = 0$. Substituting them into (2.5) yields (f). Result (g) can be shown similarly.

From (2.5), $i_+(M) = m$ is equivalent to $i_+(E_BAE_B) = m - r(B) = r(E_B)$. Also note that $i_+(E_BAE_B) \leqslant r(E_BAE_B) \leqslant r(E_B)$ always holds. These two results are further equivalent to $i_-(E_BAE_B) = 0$ and $r(E_BAE_B) = r(E_B)$. Similarly we can show that $i_-(M) = m$ is equivalent to $i_+(E_BAE_B) = 0$ and $r(E_BAE_B) = r(E_B)$. Combining the two equivalences leads to (h).

Results (i) and (j) are obvious.

The range inclusion $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ is equivalent to $BB^\dagger A = A$ by (1.2) and (1.20). Hence $E_BAE_B = 0$ follows. In this case, (2.5) reduces to the two equalities in (k).

If $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, then $r(E_BAE_B) = r(A)$ by (1.4), which is also equivalent to $i_\pm(E_BAE_B) = i_\pm(A)$ by Lemma 1.6(a). Hence (2.5) reduces to the two equalities in (l). By the similar approach, we are also able to obtain the results in (m)–(q) from (2.8) and (2.9). $\square$

Theorem 2.3 provides a group of explicit formulas for calculating inertias of block Hermitian matrices under general settings. These formulas would save a considerable amount of computation for bordered and block Hermitian matrices of large orders. Also note that these formulas are consistent with the rank formulas in (1.4)–(1.7). Hence these inertia formulas together with the rank formulas in (1.4)–(1.7) can be used to study various problems on ranks and inertias of Hermitian matrices and their applications.

Inertias of block Hermitian matrices were widely considered in the literature. Some equalities and inequalities for inertias of 2×2 block Hermitian matrices under some general settings were established. For instance, (2.5) was given in [16]; Theorem 2.3(m) was given in [10,18,21]; (2.8) was given in [31]. Some other equalities and inequalities for inertias of M and N in (2.4) and their applications can be found, e.g., in [7,10,15,16,22,28,31,34].

3. Miscellaneous applications

It is well known that the matrix expression $D - B^*A^\dagger B$ in (2.4) is called the Schur complement of A in N. From (1.9), the rank of $D - B^*A^\dagger B$ can be written as

$$r(D - B^*A^\dagger B) = r(J) - r(A), \tag{3.1}$$

where J is the following block Hermitian matrix

$$J = \begin{bmatrix} A^3 & AB \\ (AB)^* & D \end{bmatrix}. \tag{3.2}$$

As a direct consequence, we see from Lemma 1.3(a) and (3.1) that $D = B^*A^\dagger B$ if and only if $r(J) = r(A)$. Applying (1.9) and (3.1) repeatedly, we can represent the rank of any matrix expression involving Moore–Penrose inverses through ranks of some block matrices, in which no Moore–Penrose inverses are included. The rank formulas established from this process can be used to characterize properties of matrix expressions composed of Moore–Penrose inverses. In particular, they can be used to derive necessary and sufficient conditions for various matrix equalities composed of Moore–Penrose inverses to hold. Motivated by (1.1) and (3.1), we obtain the following formulas on the positive and negative signatures of the Schur complement $D - B^*A^\dagger B$.

Theorem 3.1. *Let $A \in \mathbb{C}_h^{m\times m}$, $B \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}_h^{n\times n}$. Then*

$$i_\pm(D - B^*A^\dagger B) = i_\pm(J) - i_\pm(A), \tag{3.3}$$

where J is as given in (3.2). Hence,

(a) $D - B^*A^\dagger B > 0 (< 0)$ *if and only if* $i_+(J) = n + i_+(A)$ $(i_-(J) = n + i_-(A))$.
(b) $D - B^*A^\dagger B \geqslant 0 (\leqslant 0)$ *if and only if* $i_-(J) = i_-(A)$ $(i_+(J) = i_+(A))$.

Proof. Note that $\mathcal{R}(A^3) = \mathcal{R}(A)$ for any Hermitian matrix A. Hence the range inclusion $\mathcal{R}(AB) \subseteq \mathcal{R}(A^3)$ follows. In this case, applying Theorem 2.3(m) and $A^\dagger = A(A^3)^\dagger A$ to the J in (3.2), we obtain (3.3). Applying Lemma 1.3(c) and (d) to (3.3) gives rise to (a) and (b). $\square$

Obviously, adding the two formulas in (3.3) together yields (3.1). Some useful consequences of (3.3) are given below.

Corollary 3.2. *Let* $A_1 \in \mathbb{C}_h^{m_1\times m_1}$, $A_2 \in \mathbb{C}_h^{m_2\times m_2}$, $B_1 \in \mathbb{C}^{m_1\times n}$, $B_2 \in \mathbb{C}^{m_2\times n}$ *and* $D \in \mathbb{C}_h^{n\times n}$. *Then*

$$i_{\pm}(D - B_1^*A_1^{\dagger}B_1 - B_2^*A_2^{\dagger}B_2) = i_{\pm}\begin{bmatrix} A_1^3 & 0 & A_1B_1 \\ 0 & A_2^3 & A_2B_2 \\ B_1^*A_1 & B_2^*A_2 & D \end{bmatrix} - i_{\pm}(A_1) - i_{\pm}(A_2). \tag{3.4}$$

Proof. Setting $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ in (3.3) yields (3.4). □

Corollary 3.3. *Let* $A \in \mathbb{C}_h^{m\times m}$, $B \in \mathbb{C}^{q\times n}$, $P \in \mathbb{C}^{q\times m}$ *and* $D \in \mathbb{C}_h^{n\times n}$. *Then*

$$i_{\pm}[D - B^*(P^*)^{\dagger}AP^{\dagger}B] = i_{\pm}\begin{bmatrix} -PAP^* & PP^*P & 0 \\ P^*PP^* & 0 & P^*B \\ 0 & B^*P & D \end{bmatrix} - r(P). \tag{3.5}$$

In particular, if $\mathcal{R}(A) \subseteq \mathcal{R}(P^*)$ *and* $\mathcal{R}(B) \subseteq \mathcal{R}(P)$, *then*

$$i_{\pm}[D - B^*(P^{\dagger})^*AP^{\dagger}B] = i_{\pm}\begin{bmatrix} -A & P^* & 0 \\ P & 0 & B \\ 0 & B^* & D \end{bmatrix} - r(P). \tag{3.6}$$

Proof. Applying Theorem 2.3(m), (1.13), (1.14) and (3.3) to $D - B^*(P^*)^{\dagger}AP^{\dagger}B$, and simplifying by EBCMOs gives

$$\begin{aligned}
& i_{\pm}[D - B^*(P^*)^{\dagger}AP^{\dagger}B] \\
&= i_{\pm}\begin{bmatrix} A & AP^{\dagger}B \\ B^*(P^*)^{\dagger}A & D \end{bmatrix} - i_{\pm}(A) \\
&= i_{\pm}\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix}\begin{bmatrix} 0 & P^* \\ P & 0 \end{bmatrix}^{\dagger}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) - i_{\pm}(A) \\
&= i_{\pm}\begin{bmatrix} -\begin{bmatrix} 0 & P^* \\ P & 0 \end{bmatrix}^3 & \begin{bmatrix} 0 & P^* \\ P & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \\ \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix}\begin{bmatrix} 0 & P^* \\ P & 0 \end{bmatrix} & \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \end{bmatrix} - i_{\pm}\begin{bmatrix} 0 & -P^* \\ -P & 0 \end{bmatrix} - i_{\pm}(A) \\
&= i_{\pm}\begin{bmatrix} 0 & -P^*PP^* & 0 & P^*B \\ -PP^*P & 0 & PA & 0 \\ 0 & AP^* & A & 0 \\ B^*P & 0 & 0 & D \end{bmatrix} - r(P) - i_{\pm}(A) \\
&= i_{\pm}\begin{bmatrix} 0 & -P^*PP^* & 0 & P^*B \\ -PP^*P & -PAP^* & 0 & 0 \\ 0 & 0 & A & 0 \\ B^*P & 0 & 0 & D \end{bmatrix} - r(P) - i_{\pm}(A) \\
&= i_{\pm}\begin{bmatrix} 0 & P^*PP^* & P^*B \\ PP^*P & -PAP^* & 0 \\ B^*P & 0 & D \end{bmatrix} - r(P),
\end{aligned}$$

establishing (3.5). Eq. (3.6) is derived from (3.5) and Lemma 1.6(b). □

The formulas in (3.4) and (3.5) show that the positive and negative signatures of any matrix expression involving Moore–Penrose inverses of Hermitian matrices can be represented through the positive and negative signatures of some block matrices composed of the given matrices in the expression.

Eqs. (3.3)–(3.5) and their variations can be used to derive various equalities for inertias of matrix expressions involving Moore–Penrose inverses of Hermitian matrices. In particular, these equalities for inertias can be used to characterize various inequalities for Hermitian matrices in the Löwner partial ordering. Generally speaking, necessary and sufficient conditions for any two Hermitian matrix equality and inequality

$$p(A_1^{\dagger}, \ldots, A_k^{\dagger}) = q(B_1^{\dagger}, \ldots, B_s^{\dagger}),$$
$$p(A_1^{\dagger}, \ldots, A_k^{\dagger}) \geqslant q(B_1^{\dagger}, \ldots, B_s^{\dagger})$$

to hold can always be derived from the previous rank and inertia formulas. For instance, if A is Hermitian, then it can be derived from (1.13), (3.3) and EBCMOs that

$$\begin{aligned} i_{\pm}(I_m - A^{\dagger}) &= i_{\pm}\begin{bmatrix} A^3 & A \\ A & I_m \end{bmatrix} - i_{\pm}(A) \\ &= i_{\pm}\begin{bmatrix} A^3 - A^2 & 0 \\ 0 & I_m \end{bmatrix} - i_{\pm}(A) = i_{\pm}(A^3 - A^2) + i_{\pm}(I_m) - i_{\pm}(A), \\ i_{\pm}(A - A^{\dagger}) &= i_{\pm}\begin{bmatrix} A^3 & A \\ A & A \end{bmatrix} - i_{\pm}(A) = i_{\pm}\begin{bmatrix} A^3 - A & 0 \\ 0 & A \end{bmatrix} - i_{\pm}(A) = i_{\pm}(A^3 - A), \\ i_{\pm}(A^3 - A^{\dagger}) &= i_{\pm}\begin{bmatrix} A^3 & A \\ A & A^3 \end{bmatrix} - i_{\pm}(A) = i_{\pm}(A^3 + A) + i_{\pm}(A^3 - A) - i_{\pm}(A) \quad \text{(by (2.1))}. \end{aligned}$$

In the theory of generalized inverses of matrices, Moore–Penrose inverses of matrices were primarily used to represent consistency conditions of linear matrix equations and their general solutions. It was shown by Penrose [32] that the matrix equation $AXB = C$ is solvable for X if and only if $AA^{\dagger}CB^{\dagger}B = C$. In particular, the matrix equation $P^*XP = A$ is solvable for X if and only if $P^{\dagger}PAP^{\dagger}P = A$. This fact prompts us to obtain the following result on the rank and the positive and negative signatures of the difference $A - P^{\dagger}PAP^{\dagger}P$.

Corollary 3.4. *Let $A \in \mathbb{C}_h^{m\times m}$ and $P \in \mathbb{C}^{n\times m}$. Then*

$$i_{\pm}(A - P^{\dagger}PAP^{\dagger}P) = i_{\pm}\begin{bmatrix} A & AP^* & P^* \\ PA & 0 & 0 \\ P & 0 & 0 \end{bmatrix} - r(P) = i_{\pm}\begin{bmatrix} F_PAF_P & F_PAP^* \\ PAF_P & 0 \end{bmatrix}, \tag{3.7}$$

$$r(A - P^{\dagger}PAP^{\dagger}P) = r\begin{bmatrix} A & AP^* & P^* \\ PA & 0 & 0 \\ P & 0 & 0 \end{bmatrix} - 2r(P) = r\begin{bmatrix} F_PAF_P & F_PAP^* \\ PAF_P & 0 \end{bmatrix}. \tag{3.8}$$

Hence,

(a) *$A \geqslant P^{\dagger}PAP^{\dagger}P$ if and only if $F_PAF_P \geqslant 0$ and $\mathcal{R}(AP^*) \subseteq \mathcal{R}(P^*)$.*
(b) *$A \leqslant P^{\dagger}PAP^{\dagger}P$ if and only if $F_PAF_P \leqslant 0$ and $\mathcal{R}(AP^*) \subseteq \mathcal{R}(P^*)$.*
(c) *$A = P^{\dagger}PAP^{\dagger}P$ if and only if $\mathcal{R}(A) \subseteq \mathcal{R}(P^*)$.*

Proof. Setting $D = A$ and $B = P$ in (3.5) and simplifying by EBCMOs, we obtain

$$\begin{aligned} i_{\pm}(A - P^{\dagger}PAP^{\dagger}P) &= i_{\pm}\begin{bmatrix} -PAP^* & PP^*P & 0 \\ P^*PP^* & 0 & P^*P \\ 0 & P^*P & A \end{bmatrix} - r(P) \\ &= i_{\pm}\begin{bmatrix} 0 & 0 & -PA \\ 0 & 0 & P^*P \\ -AP^* & P^*P & A \end{bmatrix} - r(P) \end{aligned}$$

$$= i_{\pm}\begin{bmatrix} 0 & 0 & PA \\ 0 & 0 & P \\ AP^* & P^* & A \end{bmatrix} - r(P) = i_{\pm}\begin{bmatrix} F_PAF_P & F_PAP^* \\ PAF_P & 0 \end{bmatrix} \text{ (by (2.5))},$$

establishing (3.7). Adding the two equalities in (3.7) leads to (3.8). Results (a)–(c) follow from (3.7), (3.8) and Lemma 1.3. □

From Theorem 2.3, we are also able to derive the following result on the positive and negative signatures of the difference of two Hermitian matrices.

Theorem 3.5. *Let $A, B \in \mathbb{C}_h^{m\times m}$. Then,*

(a) *The following equalities hold*

$$i_{\pm}(A - B) = i_{\pm}(A) - i_{\pm}(B) + i_{\pm}\begin{bmatrix} 0 & E_AB \\ BE_A & B - BA^{\dagger}B \end{bmatrix}, \tag{3.9}$$

$$i_{\pm}(A - B) = r(A - B) - i_{\mp}(A) + i_{\mp}(B) - i_{\mp}\begin{bmatrix} 0 & E_AB \\ BE_A & B - BA^{\dagger}B \end{bmatrix}, \tag{3.10}$$

$$r(A - B) = r(A) - r(B) + r\begin{bmatrix} 0 & E_AB \\ BE_A & B - BA^{\dagger}B \end{bmatrix}. \tag{3.11}$$

(b) *The following inequalities hold*

$$\max\{s_1, s_2\} \leqslant i_{\pm}(A - B) \leqslant \min\{s_3, s_4\}, \tag{3.12}$$
$$\max\{t_1, t_2\} \leqslant r(A - B) \leqslant \min\{t_3, t_4\}, \tag{3.13}$$

where

$$\begin{aligned} s_1 &= r[A, B] - i_{\mp}(A) - i_{\pm}(B), \\ s_2 &= i_{\pm}(B - BA^{\dagger}B) + i_{\pm}(A) - i_{\pm}(B), \\ s_3 &= r(A - B) + i_{\mp}(A) + i_{\mp}(B) - r[A, B], \\ s_4 &= i_{\pm}(B - BA^{\dagger}B) + r[A, B] - i_{\mp}(A) - i_{\pm}(B), \\ t_1 &= 2r[A, B] - r(A) - r(B), \\ t_2 &= r(B - BA^{\dagger}B) + r(A) - r(B), \\ t_3 &= 2r(A - B) + r(A) + r(B) - 2r[A, B], \\ t_4 &= r(B - BA^{\dagger}B) + 2r[A, B] - r(A) - r(B). \end{aligned}$$

In particular,

(c) *If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, then*

$$\begin{aligned} i_{\pm}(A - B) &= i_{\pm}(A) - i_{\pm}(B) + i_{\pm}(B - BA^{\dagger}B), \\ r(A - B) &= r(A) - r(B) + r(B - BA^{\dagger}B). \end{aligned}$$

(d) *If $BA^{\dagger}B = B$, then*

$$\begin{aligned} i_{\pm}(A - B) &= r[A, B] - i_{\mp}(A) - i_{\pm}(B), \\ r(A - B) &= 2r[A, B] - r(A) - r(B). \end{aligned}$$

(e) $i_+(A-B) = i_+(A) - i_+(B)$ *if and only if* $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ *and* $B - BA^{\dagger}B \leqslant 0$.
(f) $i_-(A-B) = i_-(A) - i_-(B)$ *if and only if* $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ *and* $B - BA^{\dagger}B \geqslant 0$.
(g) *Both* $i_+(A-B) = i_+(A) - i_+(B)$ *and* $i_-(A-B) = i_-(A) - i_-(B) \Leftrightarrow r(A-B) = r(A) - r(B)$ $\Leftrightarrow \mathcal{R}(B) \subseteq \mathcal{R}(A)$ *and* $BA^{\dagger}B = B$.
(h) *If* $A \geqslant 0$ *and* $B \geqslant 0$, *then*

$$i_-(A-B) = i_-\begin{bmatrix} 0 & E_AB \\ BE_A & B - BA^{\dagger}B \end{bmatrix}. \tag{3.14}$$

Hence $A \geqslant B$ *if and only if* $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ *and* $B - BA^{\dagger}B \geqslant 0$.

Proof. Applying (2.6) to the matrix $\begin{bmatrix} A & B \\ B & B \end{bmatrix}$ and simplifying by (1.13) yields the following equalities

$$i_{\pm}\begin{bmatrix} A & B \\ B & B \end{bmatrix} = i_{\pm}(A) + i_{\pm}\begin{bmatrix} 0 & E_AB \\ BE_A & B - BA^{\dagger}B \end{bmatrix},$$

$$i_{\pm}\begin{bmatrix} A & B \\ B & B \end{bmatrix} = i_{\pm}\begin{bmatrix} A-B & 0 \\ 0 & B \end{bmatrix} = i_{\pm}(A-B) + i_{\pm}(B).$$

Combining these equalities leads to (3.9). Adding $i_{\mp}(A-B)$ on both sides of (3.9) yields (3.10). Adding the two equalities in (3.9) leads to (3.11). Results (b)–(h) are derived from (3.9) and (3.11), and the details are omitted. □

Theorem 3.5(h) also prompts us to obtain the following formulas for the positive and negative signatures of $A^{\dagger} - B^{\dagger}$ under the assumption $A \geqslant B \geqslant 0$.

Theorem 3.6. *Assume that* $A, B \in \mathbb{C}_h^{m\times m}$ *satisfy* $A \geqslant B \geqslant 0$. *Then*

$$i_+(A^{\dagger} - B^{\dagger}) = r(A) - r(B), \tag{3.15}$$

$$i_-(A^{\dagger} - B^{\dagger}) = r(B^3 - BAB), \tag{3.16}$$

$$r(A^{\dagger} - B^{\dagger}) = r(A) - r(B) + r(B^3 - BAB). \tag{3.17}$$

Hence,

(a) $A^{\dagger} \leqslant B^{\dagger}$ *if and only if* $r(A) = r(B)$.
(b) $A^{\dagger} \geqslant B^{\dagger}$ *if and only if* $B^3 = BAB$.

Proof. Let

$$M = \begin{bmatrix} -A^3 & 0 & A \\ 0 & B^3 & B \\ A & B & 0 \end{bmatrix}. \tag{3.18}$$

Applying Theorem 2.3(m) to the M and simplifying by (2.3) and (2.4) gives

$$\begin{aligned} i_{\pm}(M) &= i_{\pm}(-A^3) + i_{\pm}(B^3) + i_{\pm}[A(A^3)^{\dagger}A - B(B^3)^{\dagger}B] \\ &= i_{\mp}(A) + i_{\pm}(B) + i_{\pm}(A^{\dagger} - B^{\dagger}). \end{aligned} \tag{3.19}$$

Also note from Theorem 3.5(h) that the inequalities $A \geqslant B \geqslant 0$ imply that $AA^{\dagger}B = B$. In this case, it is easy to verify

$$\begin{bmatrix} I_m & 0 & \frac{1}{2}A^2 \\ -BA^{\dagger} & I_m & -BA \\ 0 & 0 & I_m \end{bmatrix} M \begin{bmatrix} I_m & -A^{\dagger}B & 0 \\ 0 & I_m & 0 \\ \frac{1}{2}A^2 & -AB & I_m \end{bmatrix} = \begin{bmatrix} 0 & 0 & A \\ 0 & B^3 - BAB & 0 \\ A & 0 & 0 \end{bmatrix}.$$

Hence it follows from (1.10), (1.13) and (1.14) that

$$i_{\pm}(M) = r(A) + i_{\pm}(B^3 - BAB). \tag{3.20}$$

Combining (3.19) and (3.20) leads to

$$i_{\pm}(A^{\dagger} - B^{\dagger}) = i_{\pm}(B^3 - BAB) + i_{\pm}(A) - i_{\pm}(B). \tag{3.21}$$

Since $A \geqslant 0$, $B \geqslant 0$ and $B^3 - BAB \leqslant 0$, (3.21) reduces to (3.15) and (3.16). Results (a) and (b) are direct consequences of (3.15) and (3.16). $\square$

In what follows, we derive some equalities and inequalities for the positive and negative signatures of the matrix product BAB^*, where A is Hermitian.

Theorem 3.7. *Let $A \in \mathbb{C}_h^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$. Then,*

(a) *The following equalities hold*

$$i_{\pm}(B^*AB) = r(AB) - i_{\mp}(A) + i_{\mp}(E_{AB}AE_{AB}), \tag{3.22}$$

$$i_{\pm}(B^*AB) = r(B^*AB) - r(AB) + i_{\pm}(A) - i_{\pm}(E_{AB}AE_{AB}), \tag{3.23}$$

$$i_{\pm}(B^*AB) = i_{\pm}(A) - i_{\pm}\begin{bmatrix} A - (AB)(B^*AB)^{\dagger}(AB)^* & ABE_{(B^*AB)} \\ E_{(B^*AB)}(AB)^* & 0 \end{bmatrix}, \tag{3.24}$$

$$i_{\pm}(B^*AB) = r(B^*AB) - i_{\mp}(A) + i_{\mp}\begin{bmatrix} A - (AB)(B^*AB)^{\dagger}(AB)^* & ABE_{(B^*AB)} \\ E_{(B^*AB)}(AB)^* & 0 \end{bmatrix}, \tag{3.25}$$

$$r(B^*AB) = 2r(AB) - r(A) + r(E_{AB}AE_{AB}), \tag{3.26}$$

$$r(B^*AB) = r(A) - r\begin{bmatrix} A - (AB)(B^*AB)^{\dagger}(AB)^* & ABE_{(B^*AB)} \\ E_{(B^*AB)}(AB)^* & 0 \end{bmatrix}. \tag{3.27}$$

(b) *The following inequalities hold*

$$r(AB) - i_{\mp}(A) \leqslant i_{\pm}(B^*AB) \leqslant i_{\pm}(A) + r(B^*AB) - r(AB) \leqslant i_{\pm}(A). \tag{3.28}$$

In particular,

(c) *$r(B^*AB) = 2r(AB) - r(A)$ if and only if $E_{AB}AE_{AB} = 0$.*
(d) *$i_+(B^*AB) = r(AB) - i_-(A)$ if and only if $E_{AB}AE_{AB} \geqslant 0$.*
(e) *$i_-(B^*AB) = r(AB) - i_+(A)$ if and only if $E_{AB}AE_{AB} \leqslant 0$.*
(f) *$i_+(B^*AB) = i_+(A)$ if and only if $r(B^*AB) = r(AB)$ and $A - AB(B^*AB)^{\dagger}B^*A \leqslant 0$.*
(g) *$i_-(B^*AB) = i_-(A)$ if and only if $r(B^*AB) = r(AB)$ and $A - AB(B^*AB)^{\dagger}B^*A \geqslant 0$.*
(h) *Ref. [3] $r(B^*AB) = r(A) \Leftrightarrow i_+(B^*AB) = i_+(A) \Leftrightarrow i_-(B^*AB) = i_-(A) \Leftrightarrow r(B^*AB) = r(AB)$ and $(AB)(B^*AB)^{\dagger}(AB)^* = A$.*

Proof. Let

$$M_1 = \begin{bmatrix} -A & AB \\ (AB)^* & 0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} A & AB \\ (AB)^* & B^*AB \end{bmatrix}. \tag{3.29}$$

Applying (2.5) and (2.6) to the M_1 yields

$$i_{\pm}(M_1) = r(AB) + i_{\pm}(-E_{AB}AE_{AB}) = r(AB) + i_{\mp}(E_{AB}AE_{AB}), \tag{3.30}$$

$$i_{\pm}(M_1) = i_{\pm}(-A) + i_{\pm}(B^*AB) = i_{\mp}(A) + i_{\pm}(B^*AB). \tag{3.31}$$

Combining (3.30) and (3.31) leads to (3.22). Adding $i_{\mp}(B^*AB)$ on both sides of (3.22) yields (3.23). Adding the equalities in (3.22) yields (3.26).

Applying (2.8) to the M_2 in (3.29) gives the following equalities

$$i_{\pm}(M_2) = i_{\pm}(A) \quad \text{and} \quad i_{\pm}(M_2) = i_{\pm}(B^*AB) + i_{\pm}\begin{bmatrix} A - (AB)(B^*AB)^{\dagger}(AB)^* & ABE_{(B^*AB)} \\ E_{(B^*AB)}(AB)^* & 0 \end{bmatrix}.$$

Combining these equalities leads to (3.24). Adding $i_{\mp}(B^*AB)$ on both sides of (3.24) yields (3.25). Adding the two equalities in (3.24) yields (3.27). The inequalities in (3.28) follow from (3.22)–(3.25). Results (c)–(h) follow from (3.22)–(3.27) and Lemma 1.3. □

Ranks and inertias of sums of Hermitian matrices and their applications were considered by some authors; see, e.g., [17,27,31,33]. Note that the sum $A_1 + \cdots + A_k$ of k Hermitian matrices can be written as a triple matrix product

$$A_1 + \cdots + A_k = PNP^*,$$

where $P^* = [I_m, \ldots, I_m]$ and $N = \mathrm{diag}(A_1, \ldots, A_k)$. Applying Theorem 3.7 to the product yields the following result.

Theorem 3.8. *Let $A_1, \ldots, A_k \in \mathbb{C}_h^{m\times m}$, and denote*

$$A = A_1 + \cdots + A_k, \quad N = \mathrm{diag}(A_1, \ldots, A_k), \quad \widehat{A} = [A_1, \ldots, A_k], \quad P^* = [I_m, \ldots, I_m].$$

Then,

(a) *The following equalities hold*

$$i_{\pm}(A) = r(\widehat{A}) - i_{\mp}(A_1) - \cdots - i_{\mp}(A_k) + i_{\mp}(E_{NP}NE_{NP}), \tag{3.32}$$

$$i_{\pm}(A) = r(A) - r(\widehat{A}) + i_{\pm}(A_1) + \cdots + i_{\pm}(A_k) - i_{\pm}(E_{NP}NE_{NP}), \tag{3.33}$$

$$i_{\pm}(A) = i_{\pm}(A_1) + \cdots + i_{\pm}(A_k) - i_{\pm}\begin{bmatrix} N - (NP)(P^*NP)^{\dagger}(NP)^* & NPE_{(P^*NP)} \\ E_{(P^*NP)}(NP)^* & 0 \end{bmatrix}, \tag{3.34}$$

$$i_{\pm}(A) = r(A) - i_{\mp}(A_1) - \cdots - i_{\mp}(A_k) + i_{\mp}\begin{bmatrix} N - (NP)(P^*NP)^{\dagger}(NP)^* & NPE_{(P^*NP)} \\ E_{(P^*NP)}(NP)^* & 0 \end{bmatrix}, \tag{3.35}$$

$$r(A) = 2r(\widehat{A}) - r(A_1) - \cdots - r(A_k) + r(E_{NP}NE_{NP}), \tag{3.36}$$

$$r(A) = r(A_1) + \cdots + r(A_k) - r\begin{bmatrix} N - (NP)(P^*NP)^{\dagger}(NP)^* & NPE_{(P^*NP)} \\ E_{(P^*NP)}(NP)^* & 0 \end{bmatrix}. \tag{3.37}$$

(b) *The following inequalities hold*

$$i_{\pm}(A) \geqslant r(\widehat{A}) - i_{\mp}(A_1) - \cdots - i_{\mp}(A_k), \tag{3.38}$$

$$i_{\pm}(A) \leqslant r(A) - r(\widehat{A}) + i_{\pm}(A_1) + \cdots + i_{\pm}(A_k) \leqslant i_{\pm}(A_1) + \cdots + i_{\pm}(A_k), \tag{3.39}$$

$$r(A) \geqslant 2r(\widehat{A}) - r(A_1) - \cdots - r(A_k), \tag{3.40}$$

$$r(A) \leqslant r(A_1) + \cdots + r(A_k). \tag{3.41}$$

In particular,

(c) *$r(A) = 2r[A_1, \ldots, A_k] - r(A_1) - \cdots - r(A_k)$ if and only if $E_{NP}NE_{NP} = 0$.*
(d) *$i_+(A) = r[A_1, \ldots, A_k] - i_-(A_1) - \cdots - i_-(A_k)$ if and only if $E_{NP}NE_{NP} \geqslant 0$.*
(e) *$i_-(A) = r[A_1, \ldots, A_k] - i_+(A_1) - \cdots - i_+(A_k)$ if and only if $E_{NP}NE_{NP} \leqslant 0$.*
(f) *Both $i_+(A) = i_+(A_1) + \cdots + i_+(A_k)$ and $i_-(A) = i_-(A_1) + \cdots + i_-(A_k)$ if and only if $r(A) = r(A_1) + \cdots + r(A_k)$.*

Two Hermitian matrices A and B of the same size are said to be parallel summable if $A(A+B)^-B$ is invariant with respect to the choice of $(A+B)^-$. In this case, the parallel sum of A and B is defined to be $A : B = A(A+B)^-B = A(A+B)^{\dagger}B$. It is well known that A and B are parallel summable if and only if $r(A+B) = r[A, B]$. An alternative expression of $A : B$ is given by

$$A : B = A(A+B)^{\dagger}B = A - A(A+B)^{\dagger}A.$$

Theorem 3.9. *Assume that* $A, B \in \mathbb{C}_h^{m\times m}$ *are parallel summable. Then*

$$i_\pm(A : B) = i_\pm(A) + i_\pm(B) - i_\pm(A+B), \tag{3.42}$$
$$r(A : B) = r(A) + r(B) - r(A+B). \tag{3.43}$$

Hence,

(a) $A : B \geqslant 0$ *if and only if* $i_-(A+B) = i_-(A) + i_-(B)$.
(b) $A : B \leqslant 0$ *if and only if* $i_+(A+B) = i_+(A) + i_+(B)$.

Proof. Applying Theorem 2.3(m) to $A : B$ and simplifying by EBCMOs gives

$$\begin{aligned} i_\pm(A : B) &= i_\pm[A - A(A+B)^\dagger A] \\ &= i_\pm\begin{bmatrix} A+B & A \\ A & A \end{bmatrix} - i_\pm(A+B) \\ &= i_\pm\begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} - i_\pm(A+B) \\ &= i_\pm(A) + i_\pm(B) - i_\pm(A+B), \end{aligned}$$

establishing (3.42). Adding the two equalities in (3.42) yields (3.43). Results (a) and (b) follow from (3.42) and Lemma 1.3(d). □

Another set of formulas for the positive and negative signatures of the sum $A_1 + \cdots + A_k$ of k Hermitian matrices are given below.

Theorem 3.10. *Let* $A_1, \ldots, A_k \in \mathbb{C}_h^{m\times m}$, *and denote* $A = A_1 + \cdots + A_k$, $\widetilde{A} = \mathrm{diag}(A_1, \ldots, A_k)$ *and* $\widehat{A} = [A_1, \ldots, A_k]$. *Then,*

$$i_\pm(A) = i_\pm(A_1) + \cdots + i_\pm(A_k) - i_\pm(\widetilde{A} - \widehat{A}^*A^\dagger\widehat{A}), \tag{3.44}$$
$$r(A) = r(A_1) + \cdots + r(A_k) - r(\widetilde{A} - \widehat{A}^*A^\dagger\widehat{A}). \tag{3.45}$$

In particular,

(a) $\widetilde{A} = \widehat{A}^*A^\dagger\widehat{A}$ *if and only if* $r(A) = r(A_1) + \cdots + r(A_k)$.
(b) $\widetilde{A} \geqslant \widehat{A}^*A^\dagger\widehat{A}$ *if and only if* $i_-(A) = i_-(A_1) + \cdots + i_-(A_k)$.
(c) $\widetilde{A} \leqslant \widehat{A}^*A^\dagger\widehat{A}$ *if and only if* $i_+(A) = i_+(A_1) + \cdots + i_+(A_k)$.
(d) *If* $A_1 \geqslant 0, \ldots, A_k \geqslant 0$, *then* $\widetilde{A} \geqslant \widehat{A}^*A^\dagger\widehat{A}$.

Proof. Let $P = [I_m, \ldots, I_m]^*$. Then $\widehat{A} = P^*\widetilde{A}$ and $A = P^*\widetilde{A}P$. In this case, applying (3.1) and simplifying by (1.13) and EBCMOs gives

$$\begin{aligned} i_\pm(\widetilde{A} - \widehat{A}^*A^\dagger\widehat{A}) &= i_\pm\begin{bmatrix} A^3 & A\widehat{A} \\ \widehat{A}^*A & \widetilde{A} \end{bmatrix} - i_\pm(A) \\ &= i_\pm\begin{bmatrix} A^3 - AP^*\widehat{A}PA & 0 \\ 0 & \widetilde{A} \end{bmatrix} - i_\pm(A) \\ &= i_\pm(\widetilde{A}) - i_\pm(A) \\ &= i_\pm(A_1) + \cdots + i_\pm(A_k) - i_\pm(A), \end{aligned}$$

establishing (3.44). Adding the two equalities in (3.44) yields (3.45). Results (a)–(d) follow from (3.44) and (3.45) and Lemma 1.3. □

In addition to (2.5), we are also able to derive some formulas for the positive and negative signatures of the Hermitian matrix M in (2.4).

Theorem 3.11. *Let M be as given in (2.4), and denote the $m \times m$ and submatrix in the upper-left corner of $M^{\dagger}$ as G_1, and $n \times n$ submatrix in the lower-right corner of $M^{\dagger}$ as G_2. Then*

$$i_{\pm}(M) = i_{\pm}(A) + r(B) - i_{\pm}(A - AG_1A), \tag{3.46}$$

$$i_{\pm}(M) = r(M) - i_{\mp}(A) - r(B) + i_{\mp}(A - AG_1A), \tag{3.47}$$

$$i_{\pm}(M) = i_{\pm}(A) + r(B) - i_{\pm}[A - A(E_BAE_B)^{\dagger}A], \tag{3.48}$$

$$i_{\pm}(M) = r(M) - i_{\mp}(A) - r(B) + i_{\pm}[A - A(E_BAE_B)^{\dagger}A], \tag{3.49}$$

$$i_{\pm}(M) = r[A, B] - i_{\mp}(A + BG_2B^*), \tag{3.50}$$

$$i_{\pm}(M) = r(M) - r[A, B] + i_{\pm}(A + BG_2B^*), \tag{3.51}$$

$$r(M) = r(A) + 2r(B) - r(A - AG_1A), \tag{3.52}$$

$$r(M) = r(A) + 2r(B) - r[A - A(E_BAE_B)^{\dagger}A], \tag{3.53}$$

$$r(M) = 2r[A, B] - r(A + BG_2B^*). \tag{3.54}$$

Hence,

(a) $A \geqslant [A, 0]M^{\dagger}[A, 0]^* \Leftrightarrow A \geqslant A(E_BAE_B)^{\dagger}A \Leftrightarrow i_-(M) = i_-(A) + r(B)$.
(b) $A \leqslant [A, 0]M^{\dagger}[A, 0]^* \Leftrightarrow A \leqslant A(E_BAE_B)^{\dagger}A \Leftrightarrow i_+(M) = i_+(A) + r(B)$.
(c) $A = [A, 0]M^{\dagger}[A, 0]^* \Leftrightarrow A = A(E_BAE_B)^{\dagger}A \Leftrightarrow r(M) = r(A) + 2r(B)$.
(d) $A + [0, B]M^{\dagger}[0, B]^* \geqslant 0 \Leftrightarrow i_+(M) = r[A, B]$.
(e) $A + [0, B]M^{\dagger}[0, B]^* \leqslant 0 \Leftrightarrow i_-(M) = r[A, B]$.

Proof. Applying (3.3) to $A - AG_1A$, and simplifying by (1.13), (1.14) and EBCMOs gives

$$\begin{aligned} i_{\pm}(A - AG_1A) &= i_{\pm}(A - [A, 0]M^{\dagger}[A, 0]^*) \\ &= i_{\pm}\begin{bmatrix} M^3 & M[A, 0]^* \\ [A, 0]\,M & A \end{bmatrix} - i_{\pm}(M) \\ &= i_{\pm}\begin{bmatrix} M^3 - M\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}M & 0 \\ 0 & A \end{bmatrix} - i_{\pm}(M) \\ &= i_{\pm}\begin{bmatrix} BB^*A + ABB^* & BB^*B \\ B^*BB^* & 0 \end{bmatrix} + i_{\pm}(A) - i_{\pm}(M) \\ &= i_{\pm}\begin{bmatrix} 0 & BB^*B \\ B^*BB^* & 0 \end{bmatrix} + i_{\pm}(A) - i_{\pm}(M) \\ &= r(B) + i_{\pm}(A) - i_{\pm}(M), \end{aligned}$$

establishing (3.46) and (3.52). Adding $i_{\mp}(M)$ on both sides of (3.46) yields (3.47).

Applying (3.3), and simplifying by (2.5) and EBCMOs gives

$$\begin{aligned} i_{\pm}[A - A(E_BAE_B)^{\dagger}A] &= i_{\pm}\begin{bmatrix} (E_BAE_B)^3 & (E_BAE_B)E_BA \\ AE_B(E_BAE_B) & A \end{bmatrix} - i_{\pm}(E_BAE_B) \\ &= i_{\pm}\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} - i_{\pm}(E_BAE_B) \\ &= i_{\pm}(A) + r(B) - i_{\pm}(M), \end{aligned}$$

establishing (3.48) and (3.53). Adding $i_{\mp}(M)$ on both sides of (3.48) yields (3.49).

Applying (3.3), and simplifying by (1.14) and EBCMOs gives

$$\begin{aligned} i_{\pm}(A + BG_2B^*) &= i_{\pm}(A + [0, B]M^{\dagger}[0, B]^*) \\ &= i_{\pm}\begin{bmatrix} -M^3 & M[0, B]^* \\ [0, B]\,M & A \end{bmatrix} - i_{\pm}(-M) \\ &= i_{\pm}\begin{bmatrix} -M^3 + M\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}M & M\begin{bmatrix} A \\ B^* \end{bmatrix} \\ [A, B]\,M & A \end{bmatrix} - i_{\mp}(M) \\ &= i_{\pm}\begin{bmatrix} 0 & \begin{bmatrix} A \\ B^* \end{bmatrix} \\ [A, B] & A \end{bmatrix} - i_{\mp}(M) \\ &= i_{\pm}\begin{bmatrix} 0 & \begin{bmatrix} A \\ B^* \end{bmatrix} \\ [A, B] & 0 \end{bmatrix} - i_{\mp}(M) \\ &= r[A, B] - i_{\mp}(M), \end{aligned}$$

establishing (3.50) and (3.54). Adding $i_{\mp}(M)$ on both sides of (3.50) yields (3.51). Results (a)–(e) follow from (3.46)–(3.53) and Lemma 1.3. $\square$

In the remaining of this section, we give two formulas for the positive and negative signatures of the difference of two outer inverses of a Hermitian matrix.

Theorem 3.12. *Assume that $A, P, Q \in \mathbb{C}_h^{m\times m}$ satisfy $PAP = P$ and $QAQ = Q$. Then*

$$i_{\pm}(P - Q) = r[P, Q] - i_{\mp}(P) - i_{\pm}(Q), \tag{3.55}$$

$$r(P - Q) = 2r[P, Q] - r(P) - r(Q), \tag{3.56}$$

$$s(P - Q) = s(P) - s(Q). \tag{3.57}$$

Hence,

(a) *$P > Q$ if and only if $P \succcurlyeq 0$, $Q \preccurlyeq 0$ and $r[P, Q] = m$.*
(b) *$P < Q$ if and only if $P \preccurlyeq 0$, $Q \succcurlyeq 0$ and $r[P, Q] = m$.*
(c) *$P \succcurlyeq Q$ if and only if $r[P, Q] = i_+(P) - i_-(Q)$.*
(d) *$P \preccurlyeq Q$ if and only if $r[P, Q] = i_-(P) - i_+(Q)$.*
(e) *$P = Q$ if and only if $\mathcal{R}(P) = \mathcal{R}(Q)$.*
(f) *$s(P - Q) = 0$ if and only if $s(P) = s(Q)$.*
(g) *$r(P - Q) = r(P) - r(Q) \Leftrightarrow i_+(P - Q) = i_+(P) - i_+(Q) \Leftrightarrow i_-(P - Q) = i_-(P) - i_-(Q) \Leftrightarrow \mathcal{R}(Q) \subseteq \mathcal{R}(P)$.*

Proof. Let

$$M = \begin{bmatrix} -P & 0 & P \\ 0 & Q & Q \\ P & Q & 0 \end{bmatrix}.$$

Then it is easy to verify that

$$\begin{bmatrix} I_m & 0 & 0 \\ 0 & I_m & 0 \\ I_m & -I_m & I_m \end{bmatrix} M \begin{bmatrix} I_m & 0 & I_m \\ 0 & I_m & -I_m \\ 0 & 0 & I_m \end{bmatrix} = \begin{bmatrix} -P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & P - Q \end{bmatrix}.$$

It also follows from $PAP = P$ and $QAQ = Q$ that

$$\begin{bmatrix} I_m & 0 & \frac{1}{2}PA \\ 0 & I_m & -\frac{1}{2}QA \\ 0 & 0 & I_m \end{bmatrix} M \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_m & 0 \\ \frac{1}{2}AP & -\frac{1}{2}AQ & I_m \end{bmatrix} = \begin{bmatrix} 0 & 0 & P \\ 0 & 0 & Q \\ P & Q & 0 \end{bmatrix}.$$

Applying (1.10)–(1.14) to both equalities gives

$$i_{\pm}(M) = i_{\mp}(P) + i_{\pm}(Q) + i_{\pm}(P - Q) = r[P, Q],$$

establishing (3.55). Eqs. (3.56) and (3.57) are straightforward from (3.55). Results (a)–(g) follow from (1.20), (3.55), (3.56) and Lemma 1.3. $\square$

Rank formulas for outer inverses of matrices and their applications were considered by the present author in [35,37]. The proof of Theorem 3.12 shows that closed-form formulas for inertias of Hermitian outer inverses of Hermitian matrices can also be established by some elementary methods.

Assume that $A, B \in \mathbb{C}_h^{m\times m}$ are congruent, namely, there exists a nonsingular matrix P such that $A = PBP^*$. This equality, however, does not imply $A^{\dagger} = (P^*)^{-1}B^{\dagger}P^{-1}$ unless P is a unitary matrix. It is easy to verify by definition that both $A^{\dagger}$ and $(P^*)^{-1}B^{\dagger}P^{-1}$ are outer inverses of A. Hence, we have the following result.

Corollary 3.13. *Let $A = PBP^*$, where $B \in \mathbb{C}_h^{m\times m}$ and $P \in \mathbb{C}^{m\times m}$ is nonsingular. Then*

$$i_{\pm}[A^{\dagger} - (P^*)^{-1}B^{\dagger}P^{-1}] = r[PP^*A, A] - r(A), \tag{3.58}$$

$$r[A^{\dagger} - (P^*)^{-1}B^{\dagger}P^{-1}] = 2r[PP^*A, A] - 2r(A). \tag{3.59}$$

Hence

$$A^{\dagger} \geqslant (P^*)^{-1}B^{\dagger}P^{-1} \Leftrightarrow A^{\dagger} \leqslant (P^*)^{-1}B^{\dagger}P^{-1} \Leftrightarrow A^{\dagger} = (P^*)^{-1}B^{\dagger}P^{-1} \Leftrightarrow \mathcal{R}(PP^*A) \subseteq \mathcal{R}(A). \tag{3.60}$$

Proof. Applying (3.55) and simplifying by (1.10), (1.11), (1.24) and (1.26) gives

$$\begin{aligned} i_{\pm}[A^{\dagger} - (P^*)^{-1}B^{\dagger}P^{-1}] &= r[A^{\dagger}, (P^*)^{-1}B^{\dagger}P^{-1}] - i_{\mp}(A^{\dagger}) - i_{\pm}[(P^*)^{-1}B^{\dagger}P^{-1}] \\ &= r[P^*A, B] - i_{\mp}(A) - i_{\pm}(A) \\ &= r[PP^*A, A] - r(A), \end{aligned}$$

establishing (3.58) and (3.59). Setting right-hand sides of (3.58) and (3.59) equal to zero and applying Lemma 1.3 and (1.10) leads to the equivalences in (3.60). $\square$

Matrix expressions with symmetric patterns that involve Moore–Penrose inverses occur widely in matrix theory and applications. In particular, the real symmetric matrix expressions $P(P^TAP)^{\dagger}P^T$ and $AP(P^TAP)^{\dagger}P^TA$ are often used in regression analysis. The following result can be used to characterize relations between two Hermitian matrix expressions of this kind.

Corollary 3.14. *Let $A \in \mathbb{C}_h^{m\times m}$, $P \in \mathbb{C}^{m\times n}$ and $Q \in \mathbb{C}^{m\times k}$. Then,*

(a) *The following equalities*

$$i_{\pm}[P(P^*AP)^{\dagger}P^* - Q(Q^*AQ)^{\dagger}Q^*] = r[PP^*AP, QQ^*AQ] - i_{\mp}(P^*AP) - i_{\pm}(Q^*AQ), \tag{3.61}$$

$$r[P(P^*AP)^{\dagger}P^* - Q(Q^*AQ)^{\dagger}Q^*] = 2r[PP^*AP, QQ^*AQ] - r(P^*AP) - r(Q^*AQ), \tag{3.62}$$

$$i_{\pm}[AP(P^*AP)^{\dagger}P^*A - AQ(Q^*AQ)^{\dagger}Q^*A] = r[AP, AQ] - i_{\mp}(P^*AP) - i_{\pm}(Q^*AQ), \tag{3.63}$$

$$r[AP(P^*AP)^{\dagger}P^*A - AQ(Q^*AQ)^{\dagger}Q^*A] = 2r[AP, AQ] - r(P^*AP) - r(Q^*AQ) \tag{3.64}$$

hold. Hence,

(i) $P(P^*AP)^{\dagger}P^* > Q(Q^*AQ)^{\dagger}Q^*$ *if and only if*

$$P^*AP \geqslant 0, \quad Q^*AQ \leqslant 0, \quad r[PP^*AP, QQ^*AQ] = m.$$

(ii) $P(P^*AP)^{\dagger}P^* < Q(Q^*AQ)^{\dagger}Q^*$ *if and only if*

$$P^*AP \leqslant 0, \quad Q^*AQ \geqslant 0, \quad r[PP^*AP, QQ^*AQ] = m.$$

(iii) $P(P^*AP)^{\dagger}P^* \geqslant Q(Q^*AQ)^{\dagger}Q^*$ *if and only if*

$$r[PP^*AP, QQ^*AQ] = i_+(P^*AP) + i_-(Q^*AQ).$$

(iv) $P(P^*AP)^{\dagger}P^* \leqslant Q(Q^*AQ)^{\dagger}Q^*$ *if and only if*

$$r[PP^*AP, QQ^*AQ] = i_-(P^*AP) + i_+(Q^*AQ).$$

(v) $P(P^*AP)^{\dagger}P^* = Q(Q^*AQ)^{\dagger}Q^*$ *if and only if* $\mathcal{R}(PP^*AP) = \mathcal{R}(QQ^*AQ)$.

(vi) $AP(P^*AP)^{\dagger}P^*A \geqslant AQ(Q^*AQ)^{\dagger}Q^*A$ *if and only if*

$$r[AP, AQ] = i_+(P^*AP) + i_-(Q^*AQ).$$

(vii) $AP(P^*AP)^{\dagger}P^*A \leqslant AQ(Q^*AQ)^{\dagger}Q^*A$ *if and only if*

$$r[AP, AQ] = i_-(P^*AP) + i_+(Q^*AQ).$$

(viii) $AP(P^*AP)^{\dagger}P^*A = AQ(Q^*AQ)^{\dagger}Q^*A$ *if and only if*

$$2r[AP, AQ] = r(P^*AP) + r(Q^*AQ).$$

(b) *Under the condition* $A \geqslant 0$,

$$i_+[P(P^*AP)^{\dagger}P^* - Q(Q^*AQ)^{\dagger}Q^*] = r[PP^*A, QQ^*A] - r(AQ), \tag{3.65}$$

$$i_-[P(P^*AP)^{\dagger}P^* - Q(Q^*AQ)^{\dagger}Q^*] = r[PP^*A, QQ^*A] - r(AP), \tag{3.66}$$

$$i_+[AP(P^*AP)^{\dagger}P^*A - AQ(Q^*AQ)^{\dagger}Q^*A] = r[AP, AQ] - r(AQ), \tag{3.67}$$

$$i_-[AP(P^*AP)^{\dagger}P^*A - AQ(Q^*AQ)^{\dagger}Q^*A] = r[AP, AQ] - r(AP). \tag{3.68}$$

Hence,

(i) $P(P^*AP)^{\dagger}P^* \geqslant Q(Q^*AQ)^{\dagger}Q^*$ *if and only if* $r[PP^*A, QQ^*A] = r(AP)$, *i.e.,* $\mathcal{R}(QQ^*A) \subseteq \mathcal{R}(PP^*A)$.

(ii) $P(P^*AP)^{\dagger}P^* \leqslant Q(Q^*AQ)^{\dagger}Q^*$ *if and only if* $r[PP^*A, QQ^*A] = r(AQ)$, *i.e.,* $\mathcal{R}(PP^*A) \subseteq \mathcal{R}(QQ^*A)$.

(iii) $P(P^*AP)^{\dagger}P^* = Q(Q^*AQ)^{\dagger}Q^*$ *if and only if* $\mathcal{R}(PP^*A) = \mathcal{R}(QQ^*A)$.

(iv) $AP(P^*AP)^{\dagger}P^*A \geqslant AQ(Q^*AQ)^{\dagger}Q^*A$ *if and only if* $r[AP, AQ] = r(AP)$, *i.e.,* $\mathcal{R}(AQ) \subseteq \mathcal{R}(AP)$.

(v) $AP(P^*AP)^{\dagger}P^*A \leqslant AQ(Q^*AQ)^{\dagger}Q^*A$ *if and only if* $r[AP, AQ] = r(AQ)$, *i.e.,* $\mathcal{R}(AP) \subseteq \mathcal{R}(AQ)$.

(vi) $AP(P^*AP)^{\dagger}P^*A = AQ(Q^*AQ)^{\dagger}Q^*A$ *if and only if* $\mathcal{R}(AP) = \mathcal{R}(AQ)$.

Proof. It is easy to verify by definition that both $P(P^*AP)^{\dagger}P^*$ and $Q(Q^*AQ)^{\dagger}Q^*$ are outer inverses of A, while both $AP(P^*AP)^{\dagger}P^*A$ and $AQ(Q^*AQ)^{\dagger}Q^*A$ are outer inverses of $A^{\dagger}$. In these cases, applying (3.55) gives

$$\begin{aligned} & i_{\pm}[P(P^*AP)^{\dagger}P^* - Q(Q^*AQ)^{\dagger}Q^*] \\ & \quad = r[P(P^*AP)^{\dagger}P^*, Q(Q^*AQ)^{\dagger}Q^*] - i_{\mp}[P(P^*AP)^{\dagger}P^*] - i_{\pm}[Q(Q^*AQ)^{\dagger}Q^*], \end{aligned} \tag{3.69}$$

$$\begin{aligned} & i_{\pm}[AP(P^*AP)^{\dagger}P^*A - AQ(Q^*AQ)^{\dagger}Q^*A] \\ & \quad = r[AP(P^*AP)^{\dagger}P^*A, AQ(Q^*AQ)^{\dagger}Q^*A] - i_{\mp}[AP(P^*AP)^{\dagger}P^*A] - i_{\pm}[AQ(Q^*AQ)^{\dagger}Q^*A]. \end{aligned} \tag{3.70}$$

It is also easy to verify by Lemma 1.6(b) and (1.23)–(1.26) that

$$r[P(P^*AP)^{\dagger}P^*, Q(Q^*AQ)^{\dagger}Q^*] = r[PP^*AP, QQ^*AQ],$$
$$r[AP(P^*AP)^{\dagger}P^*A, AQ(Q^*AQ)^{\dagger}Q^*A] = r[AP, AQ],$$
$$i_{\pm}[P(P^*AP)^{\dagger}P^*] = i_{\pm}[AP(P^*AP)^{\dagger}P^*A] = i_{\pm}(P^*AP),$$
$$i_{\pm}[Q(Q^*AQ)^{\dagger}Q^*] = i_{\pm}[AQ(Q^*AQ)^{\dagger}Q^*A] = i_{\pm}(Q^*AQ).$$

Substituting these equalities into (3.69) and (3.70), and simplifying yields (3.61)–(3.64). Under $A \geqslant 0$, both $P^*AP \geqslant 0$ and $Q^*AQ \geqslant 0$. In these cases, $\mathcal{R}(P^*AP) = \mathcal{R}(P^*A)$, $\mathcal{R}(Q^*AQ) = \mathcal{R}(Q^*A)$, $r(P^*AP) = r(AP)$ and $r(Q^*AQ) = r(AQ)$ hold. Thus (3.61)–(3.64) reduce to (3.65)–(3.68). Results (i)–(viii) in Part (a) and (i)–(vi) in Part (b) follow from Lemma 1.3, (1.20), (3.61)–(3.64) and (3.65)–(3.68). □

Setting $P = I_m$ in Corollary 3.14 yields the following result.

Corollary 3.15. *Let $A \in \mathbb{C}_h^{m\times m}$ and $Q \in \mathbb{C}^{m\times k}$. Then,*

(a) *The following equalities*

$$i_{\pm}[A^{\dagger} - Q(Q^*AQ)^{\dagger}Q^*] = r[A, QQ^*AQ] - i_{\mp}(A) - i_{\pm}(Q^*AQ), \tag{3.71}$$
$$r[A^{\dagger} - Q(Q^*AQ)^{\dagger}Q^*] = 2r[A, QQ^*AQ] - r(A) - r(Q^*AQ), \tag{3.72}$$
$$i_{\pm}[A - AQ(Q^*AQ)^{\dagger}Q^*A] = i_{\pm}(A) - i_{\pm}(Q^*AQ), \tag{3.73}$$
$$r[A - AQ(Q^*AQ)^{\dagger}Q^*A] = r(A) - r(Q^*AQ) \tag{3.74}$$

hold. Hence,

(i) $A^{\dagger} > Q(Q^*AQ)^{\dagger}Q^*$ *if and only if* $A \geqslant 0$, $Q^*AQ \leqslant 0$ *and* $r[A, QQ^*AQ] = m$.
(ii) $A^{\dagger} < Q(Q^*AQ)^{\dagger}Q^*$ *if and only if* $A \leqslant 0$, $Q^*AQ \geqslant 0$ *and* $r[A, QQ^*AQ] = m$.
(iii) $A^{\dagger} \geqslant Q(Q^*AQ)^{\dagger}Q^*$ *if and only if* $r[A, QQ^*AQ] = i_+(A) + i_-(Q^*AQ)$.
(iv) $A^{\dagger} \leqslant Q(Q^*AQ)^{\dagger}Q^*$ *if and only if* $r[A, QQ^*AQ] = i_-(A) + i_+(Q^*AQ)$.
(v) $A^{\dagger} = Q(Q^*AQ)^{\dagger}Q^*$ *if and only if* $\mathcal{R}(A) = \mathcal{R}(QQ^*AQ)$.
(vi) $A \geqslant AQ(Q^*AQ)^{\dagger}Q^*A$ *if and only if* $i_-(A) = i_-(Q^*AQ)$.
(vii) $A \leqslant AQ(Q^*AQ)^{\dagger}Q^*A$ *if and only if* $i_+(A) = i_+(Q^*AQ)$.
(viii) $A = AQ(Q^*AQ)^{\dagger}Q^*A$ *if and only if* $r(A) = r(Q^*AQ)$.

(b) *Under the condition $A \geqslant 0$, the following equalities*

$$i_+[A^{\dagger} - Q(Q^*AQ)^{\dagger}Q^*] = r[A, QQ^*A] - r(AQ), \tag{3.75}$$
$$i_-[A^{\dagger} - Q(Q^*AQ)^{\dagger}Q^*] = r[A, QQ^*A] - r(A), \tag{3.76}$$
$$i_+[A - AQ(Q^*AQ)^{\dagger}Q^*A] = r(A) - r(AQ), \tag{3.77}$$
$$i_-[A - AQ(Q^*AQ)^{\dagger}Q^*A] = 0 \tag{3.78}$$

hold. Hence,

(i) $A^{\dagger} \geqslant Q(Q^*AQ)^{\dagger}Q^*$ *if and only if* $\mathcal{R}(QQ^*A) \subseteq \mathcal{R}(A)$.
(ii) $A^{\dagger} = Q(Q^*AQ)^{\dagger}Q^*$ *if and only if* $\mathcal{R}(A) = \mathcal{R}(QQ^*A)$.
(iii) $A \geqslant AQ(Q^*AQ)^{\dagger}Q^*A$ *always holds.*

Eq. (3.73) was given in Lemma 1 of [30]. Result (i) in Theorem 3.15(b) was given in Corollary 3 of [2]. Finally, setting $A = I_m$ in Theorem 3.12 leads to the following result.

Corollary 3.16. *Assume that* $P, Q \in \mathbb{C}^{m\times m}$ *are two orthogonal projectors, i.e.,* $P^2 = P = P^*$ *and* $Q^2 = Q = Q^*$. *Then*

$$i_+(P - Q) = r[P, Q] - r(Q), \tag{3.79}$$

$$i_-(P - Q) = r[P, Q] - r(P), \tag{3.80}$$

$$r(P - Q) = 2r[P, Q] - r(P) - r(Q), \tag{3.81}$$

$$s(P - Q) = r(P) - r(Q). \tag{3.82}$$

Hence,

(a) $P - Q$ *is nonsingular if and only if* $r[P, Q] = r(P) + r(Q) = m$.
(b) $P = Q$ *if and only if* $\mathcal{R}(P) = \mathcal{R}(Q)$.
(c) $P > Q$ $(P < Q)$ *if and only if* $P = I_m$ *and* $Q = 0$ $(P = 0$ *and* $Q = I_m)$.
(d) $P \geqslant Q$ $(P \leqslant Q)$ *if and only if* $\mathcal{R}(Q) \subseteq \mathcal{R}(P)$ $(\mathcal{R}(P) \subseteq \mathcal{R}(Q))$.
(e) $r(P - Q) = r(P) + r(Q) \Leftrightarrow i_+(P - Q) = r(P) \Leftrightarrow i_-(P - Q) = r(Q) \Leftrightarrow \mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$.
(f) $r(P - Q) = r(P) - r(Q) \Leftrightarrow i_+(P - Q) = r(P) - r(Q) \Leftrightarrow i_-(P - Q) = 0 \Leftrightarrow \mathcal{R}(Q) \subseteq \mathcal{R}(P)$.

Further work on inertias of linear combinations of orthogonal projectors and their applications will be given in another paper.

4. Extremal values of the rank and inertia of $A - BXB^*$

Let

$$p(X) = A - BXB^* \tag{4.1}$$

be a matrix expression, where $A \in \mathbb{C}_h^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$ are given, and $X \in \mathbb{C}_h^{n\times n}$ is a variable matrix. When the matrix X runs over $\mathbb{C}_h^{n\times n}$, the rank and inertia of $p(X)$ may vary with respect to the choice of X. In [38], the maximal and minimal ranks of $p(X)$ with respect to $X \in \mathbb{C}_h^{n\times n}$ were derived through generalized inverses of matrices and block matrices. Now we are able to derive the maximal and minimal possible positive and negative signatures of $p(X)$ with respect to $X \in \mathbb{C}_h^{n\times n}$. To do so, we need some results on Hermitian solutions of matrix equations.

Lemma 4.1. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}_h^{m\times m}$ *be given. Then,*

(a) [19] *The matrix equation* $AXA^* = B$ *has a solution* $X \in \mathbb{C}_h^{n\times n}$ *if and only if* $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, *or equivalently,* $AA^\dagger B = B$.
(b) *Under* $AA^\dagger B = B$, *the general Hermitian solution of* $AXA^* = B$ *can be written in the following two forms*

$$X = A^\dagger B(A^\dagger)^* + U - A^\dagger AUA^\dagger A, \tag{4.2}$$

$$X = A^\dagger B(A^\dagger)^* + F_A V + V^* F_A, \tag{4.3}$$

where $U \in \mathbb{C}_h^{n\times n}$ *and* $V \in \mathbb{C}^{n\times n}$ *are arbitrary.*

Proof. If the equation $AXA^* = B$ has a Hermitian solution, then $B = AXA^* = AA^\dagger AXA^* = AA^\dagger B$. Obviously the two equalities $AA^\dagger B = B$ and $BAA^\dagger = B$ are equivalent. In this case, substituting the two Hermitian matrices in (4.2) and (4.3) into AXA^* gives

$$AXA^* = AA^\dagger B(A^\dagger)^* A^* + AUA^* - AUA^* = BAA^\dagger = B,$$

$$AXA^* = AA^\dagger B(A^\dagger)^* A^* + AF_A V + V^* F_A A^* = BAA^\dagger = B.$$

Hence (4.2) and (4.3) are Hermitian solutions to $AXA^* = B$. Also assume that $AX_0A^* = B$ for some $X_0 \in \mathbb{C}_h^{n\times n}$. Then $A^\dagger AX_0A^\dagger A = A^\dagger B(A^\dagger)^*$. In this case, set $U = X_0$ and $V = \frac{1}{2}X_0(I_n + A^\dagger A)$. Then both (4.2) and (4.3) become

$$X = A^\dagger B(A^\dagger)^* + X_0 - A^\dagger X_0 A^\dagger A = X_0,$$

and

$$\begin{aligned} X &= A^\dagger B(A^\dagger)^* + \frac{1}{2}F_AX_0(I_n + A^\dagger A) + \frac{1}{2}(I_n + A^\dagger A)X_0F_A \\ &= A^\dagger B(A^\dagger)^* + \frac{1}{2}(X_0 + X_0A^\dagger A - A^\dagger AX_0 - A^\dagger AX_0A^\dagger A + X_0 + A^\dagger AX_0 - X_0A^\dagger A - A^\dagger AX_0A^\dagger A) \\ &= A^\dagger B(A^\dagger)^* + X_0 - A^\dagger AX_0A^\dagger A \\ &= X_0. \end{aligned}$$

These two results imply that any Hermitian solution of $AXA^* = B$ can be represented by (4.2) and (4.3). Thus (4.2) and (4.3) are the general Hermitian solutions to $AXA^* = B$. □

Lemma 4.2. *Let $A \in \mathbb{C}^{m\times n}$ be given. Then,*

(a) *The general expression of $X \in \mathbb{C}_h^{n\times n}$ satisfying the inequality $AXA^* \geqslant 0$ can be written in the following two forms*

$$X = UU^* + V - A^\dagger AVA^\dagger A, \tag{4.4}$$
$$X = UU^* + F_AW + W^*F_A, \tag{4.5}$$

where $U \in \mathbb{C}^{n\times k}$, $V \in \mathbb{C}_h^{n\times n}$ and $W \in \mathbb{C}^{n\times n}$ are arbitrary.

(b) *The general expression of $X \in \mathbb{C}_h^{n\times n}$ satisfying the inequality $AXA^* \leqslant 0$ can be written in the following two forms*

$$X = -UU^* + V - A^\dagger AVA^\dagger A, \tag{4.6}$$
$$X = -UU^* + F_AW + W^*F_A, \tag{4.7}$$

where $U \in \mathbb{C}^{n\times k}$, $V \in \mathbb{C}_h^{n\times n}$ and $W \in \mathbb{C}^{n\times n}$ are arbitrary.

Proof. Substituting (4.4) and (4.5) into AXA^* gives

$$AXA^* = AUU^*A^* + AVA^* - AA^\dagger AVA^\dagger AA^* = AUU^*A^* \geqslant 0,$$
$$AXA^* = AUU^*A^* + AF_AV + AV^*F_AA^* = AUU^*A^* \geqslant 0.$$

Hence both (4.4) and (4.5) are Hermitian solutions to the inequality $AXA^* \geqslant 0$. Also assume that X_0 is any Hermitian matrix satisfying $AX_0A^* \geqslant 0$. Then $A^\dagger AX_0A^*(A^\dagger)^* = A^\dagger AX_0A^\dagger A \geqslant 0$. Now set $UU^* = A^\dagger AX_0A^\dagger A$, $V = X_0$ and $W = \frac{1}{2}X_0(I_n + A^\dagger A)$. Then both (4.4) and (4.5) become

$$X = A^\dagger AX_0A^\dagger A + X_0 - A^\dagger AX_0A^\dagger A = X_0,$$

$$\begin{aligned} X &= A^\dagger AX_0A^\dagger A + \frac{1}{2}F_AX_0(I_n + A^\dagger A) + \frac{1}{2}X_0(I_n + A^\dagger A)F_A \\ &= A^\dagger AX_0A^\dagger A + \frac{1}{2}(X_0 + X_0A^\dagger A - A^\dagger AX_0 - A^\dagger AX_0A^\dagger A + X_0 + A^\dagger AX_0 - X_0A^\dagger A - A^\dagger AX_0A^\dagger A) \\ &= X_0. \end{aligned}$$

These two results indicate that any Hermitian solution to the inequality $AXA^* \geqslant 0$ can be represented by (4.4) and (4.5), respectively. Hence, (4.4) and (4.5) are the general solutions to $AXA^* \geqslant 0$, respectively. Result (b) can be shown similarly. $\square$

Using the matrices in (4.1), we construct a block Hermitian matrix as follows

$$N = \begin{bmatrix} A & B & 0 \\ B^* & 0 & I_n \\ 0 & I_n & -X \end{bmatrix}. \tag{4.8}$$

Applying (2.9) to this N gives

$$i_{\pm}(N) = n + r[A, B] - i_{\mp}(M) + i_{\mp}[F_{S_1}(X + S^*M^{\dagger}S)F_{S_1}], \tag{4.9}$$

$$r(N) = 2n + 2r[A, B] - r(M) + r[F_{S_1}(X + S^*M^{\dagger}S)F_{S_1}], \tag{4.10}$$

where

$$M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \quad S_1 = S - MM^{\dagger}S. \tag{4.11}$$

On the other hand,

$$\begin{bmatrix} I_m & -\frac{1}{2}BX & 0 \\ 0 & I_n & 0 \\ 0 & \frac{1}{2}X & I_n \end{bmatrix} \begin{bmatrix} A & B & 0 \\ B^* & 0 & I_n \\ 0 & I_n & -X \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ -\frac{1}{2}XB^* & I_n & \frac{1}{2}X \\ 0 & 0 & I_n \end{bmatrix} = \begin{bmatrix} A - BXB^* & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix}.$$

Applying (1.10), (1.13) and (1.14) to this equality gives the following three formulas

$$i_{\pm}(N) = n + i_{\pm}(A - BXB^*) \quad \text{and} \quad r(N) = 2n + r(A - BXB^*). \tag{4.12}$$

Combining (4.9), (4.10) and (4.12) leads to the following result.

Theorem 4.3. *Let $p(X)$ be as given in (4.1) and let M, S and S_1 be as given in (4.11). Then*

$$i_{\pm}[p(X)] = r[A, B] - i_{\mp}(M) + i_{\mp}[F_{S_1}(X + S^*M^{\dagger}S)F_{S_1}], \tag{4.13}$$

$$r[p(X)] = 2r[A, B] - r(M) + r[F_{S_1}(X + S^*M^{\dagger}S)F_{S_1}]. \tag{4.14}$$

Eq. (4.14) was given in [38]. Notice that $F_{S_1}(X + S^*M^{\dagger}S)F_{S_1}$ is a homogeneous Hermitian matrix expression with respect to $Y = X + S^*M^{\dagger}S$. Hence its extremal ranks and inertias with respect to X can easily be derived.

Theorem 4.4. *Let $p(X)$ be as given in (4.1), and let M, S and S_1 be as given in (4.11). Then*

$$\max_{X \in \mathbb{C}_h^{n\times n}} r[p(X)] = r[A, B], \tag{4.15}$$

$$\min_{X \in \mathbb{C}_h^{n\times n}} r[p(X)] = 2r[A, B] - r(M), \tag{4.16}$$

$$\max_{X \in \mathbb{C}_h^{n\times n}} i_{\pm}[p(X)] = i_{\pm}(M), \tag{4.17}$$

$$\min_{X \in \mathbb{C}_h^{n\times n}} i_{\pm}[p(X)] = r[A, B] - i_{\mp}(M). \tag{4.18}$$

(a) [38] *The general expression of $X \in \mathbb{C}_h^{n\times n}$ satisfying (4.15) can be written as*

$$X = -S^*M^{\dagger}S + U, \tag{4.19}$$

where $U \in \mathbb{C}_h^{n\times n}$ is chosen such that $r(F_{S_1}UF_{S_1}) = r(F_{S_1})$, say, $U = I_n$.

(b) *The general expressions of $X \in \mathbb{C}_h^{n\times n}$ satisfying $i_+[p(X)] = i_+(M)$ can be written as*

$$X = -S^*M^\dagger S - UU^*, \tag{4.20}$$

where $U \in \mathbb{C}^{n\times k}$ *is chosen such that* $i_-(-F_{S_1}UU^*F_{S_1}) = r(F_{S_1})$, *say,* $U = I_n$; *the general expression of* $X \in \mathbb{C}_h^{n\times n}$ *satisfying* $i_-[p(X)] = i_-(M)$ *can be written as*

$$X = -S^*M^\dagger S + UU^*, \tag{4.21}$$

where $U \in \mathbb{C}^{n\times k}$ *is chosen such that* $i_+(F_{S_1}UU^*F_{S_1}) = r(F_{S_1})$, *say,* $U = I_n$.

(c) [38] *The general expression of* $X \in \mathbb{C}_h^{n\times n}$ *satisfying* (4.16) *can be written as*

$$X = -S^*M^\dagger S + S_1^*V^* + VS_1, \tag{4.22}$$

where $V \in \mathbb{C}^{n\times(m+n)}$ *is arbitrary.*

(d) *The general expression of* $X \in \mathbb{C}_h^{n\times n}$ *satisfying* $i_+(A - BXB^*) = r[A, B] - i_-(M)$ *can be written as*

$$X = -S^*M^\dagger S - UU^* + V - F_{S_1}VF_{S_1}, \tag{4.23}$$

where $U \in \mathbb{C}^{n\times k}$ *and* $V \in \mathbb{C}_h^{n\times n}$ *are arbitrary.*

(e) *The general expression of* $X \in \mathbb{C}_h^{n\times n}$ *satisfying* $i_-[p(X)] = r[A, B] - i_+(M)$ *can be written as*

$$X = -S^*M^\dagger S + UU^* + V - F_{S_1}VF_{S_1}, \tag{4.24}$$

where $U \in \mathbb{C}^{n\times k}$ *and* $V \in \mathbb{C}_h^{n\times n}$ *are arbitrary.*

In particular,

(f) *There exists an* $X \in \mathbb{C}_h^{n\times n}$ *such that*

$$A - BXB^* > 0 \tag{4.25}$$

if and only if $i_+(M) = m$, *or equivalently* $i_-(E_BAE_B) = 0$ *and* $r(E_BAE_B) = r(E_B)$. *In this case, the general expression of* $X \in \mathbb{C}_h^{n\times n}$ *satisfying the inequality can be written as* (4.20).

(g) *There exists an* $X \in \mathbb{C}_h^{n\times n}$ *such that*

$$A - BXB^* < 0 \tag{4.26}$$

if and only if $i_-(M) = m$, *or equivalently* $i_+(E_BAE_B) = 0$ *and* $r(E_BAE_B) = r(E_B)$. *In this case, the general expression of* $X \in \mathbb{C}_h^{n\times n}$ *satisfying the inequality can be written as* (4.21).

(h) *There exists an* $X \in \mathbb{C}_h^{n\times n}$ *such that*

$$A - BXB^* \leqslant 0 \tag{4.27}$$

if and only if $i_-(M) = r[A, B]$. *In this case, the general expression* $X \in \mathbb{C}_h^{n\times n}$ *satisfying* (4.27) *can be written as in* (4.24).

(i) *There exists an* $X \in \mathbb{C}_h^{n\times n}$ *such that*

$$A - BXB^* \geqslant 0 \tag{4.28}$$

if and only if $i_+(M) = r[A, B]$. *In this case, the general expression of* $X \in \mathbb{C}_h^{n\times n}$ *satisfying* (4.28) *can be written as in* (4.23).

(j) $A - BXB^*$ *is nonsingular for all* $X \in \mathbb{C}_h^{n\times n}$ *if and only if* $r(M) = r(A) = m$.

(k) $A - BXB^* = 0$ *for all* $X \in \mathbb{C}_h^{n\times n}$ *if and only if both* $A = 0$ *and* $B = 0$.

(l) $A - BXB^* > 0$ $(A - BXB^* < 0)$ *for all* $X \in \mathbb{C}_h^{n\times n}$ *if and only if* $A > 0$ *and* $B = 0$ $(A < 0$ *and* $B = 0)$.

(m) $A - BXB^* \geqslant 0$ $(A - BXB^* \leqslant 0)$ *for all* $X \in \mathbb{C}_h^{n\times n}$ *if and only if* $A \geqslant 0$ *and* $B = 0$ $(A \leqslant 0$ *and* $B = 0)$.

(n) $r(A - BXB^*) = r(A)$ *for all* $X \in \mathbb{C}_h^{n\times n}$ $\Leftrightarrow$ $i_+(A - BXB^*) = i_+(A)$ *for all* $X \in \mathbb{C}_h^{n\times n}$ $\Leftrightarrow$ $i_-(A - BXB^*) = i_-(A)$ *for all* $X \in \mathbb{C}_h^{n\times n}$ $\Leftrightarrow$ $r(M) = r[A, B]$.

Proof. Eqs. (4.15) and (4.16) were given in [38]. It follows from (4.13) that

$$\max_{X\in\mathbb{C}_h^{n\times n}} i_\pm(A - BXB^*) = r[A, B] - i_\mp(M) + \max_{X\in\mathbb{C}_h^{n\times n}} i_\mp[F_{S_1}(X + S^*M^\dagger S)F_{S_1}], \tag{4.29}$$

$$\min_{X\in\mathbb{C}_h^{n\times n}} i_\pm(A - BXB^*) = r[A, B] - i_\mp(M) + \min_{X\in\mathbb{C}_h^{n\times n}} i_\mp[F_{S_1}(X + S^*M^\dagger S)F_{S_1}]. \tag{4.30}$$

Since F_{S_1} is Hermitian, we have

$$\max_{X \in \mathbb{C}_h^{n\times n}} i_{\mp}[F_{S_1}(X + S^*M^{\dagger}S)F_{S_1}] = r(F_{S_1}), \tag{4.31}$$

and the Hermitian matrices Xs satisfying (4.31) can be written, for example, as $X + S^*M^{\dagger}S = \mp I_n$. It can also be derived from (1.2) and (1.3) that

$$r(F_{S_1}) = n - r(E_M S) = n + r(M) - r[M, S] = r(M) - r[A, B]. \tag{4.32}$$

Substituting (4.32) into (4.31), and (4.31) into (4.29) gives

$$\max_{X \in \mathbb{C}_h^{n\times n}} i_{\pm}(A - BXB^*) = r(M) - i_{\mp}(M) = i_{\pm}(M),$$

as requited for (4.17). Also note that

$$\min_{X \in \mathbb{C}_h^{n\times n}} i_{\mp}[F_{S_1}(X + S^*M^{\dagger}S)F_{S_1}] = 0, \tag{4.33}$$

and a Hermitian matrix X satisfying (4.33) can be written, for example, as $X + S^*M^{\dagger}S = 0$. Hence we have (4.18). Solving (4.33) for X by Lemma 4.2 gives (4.23) and (4.24), respectively. Results (f)–(n) follow from (4.15)–(4.18) and Lemma 1.4. $\square$

In the investigation of the matrix equation $BXB^* = A \geqslant 0$, much attention was paid to the existence of nonnegative definite solution X to the equation; see, e.g., [1,11,20]. Motivated by this work, we obtain the following result on the extremal values of the ranks and inertias of matrix expression $A - BXB^*$ with respect to $X \geqslant 0$, where $A \geqslant 0$.

Corollary 4.5. *Let $p(X)$, M, S and S_1 be as given in (4.1) and (4.11), and assume $A \geqslant 0$. Then,*

(a) *The maximal rank of $p(X)$ subject to $X \geqslant 0$ is*

$$\max_{X \geqslant 0} r(A - BXB^*) = r[A, B], \tag{4.34}$$

and a matrix $X \geqslant 0$ satisfying (4.34) can be written as $X = -S^*M^{\dagger}S + U$, where $U > 0$.

(b) *The minimal rank of $p(X)$ subject to $X \geqslant 0$ is*

$$\min_{X \geqslant 0} r(A - BXB^*) = r[A, B] - r(B), \tag{4.35}$$

and a matrix $X \geqslant 0$ satisfying (4.35) can be written as $X = -S^*M^{\dagger}S$.

(c) *The maximal positive signature of $p(X)$ subject to $X \geqslant 0$ is*

$$\max_{X \geqslant 0} i_{+}(A - BXB^*) = r(A), \tag{4.36}$$

and a matrix $X \geqslant 0$ satisfying (4.36) is $X = 0$.

(d) *The minimal positive signature of $p(X)$ subject to $X \geqslant 0$ is*

$$\min_{X \geqslant 0} i_{+}(A - BXB^*) = r[A, B] - r(B), \tag{4.37}$$

and any $X \geqslant -S^*M^{\dagger}S$ satisfies (4.37).

(e) *The maximal negative signature of $p(X)$ subject to $X \geqslant 0$ is*

$$\max_{X \geqslant 0} i_{-}(A - BXB^*) = r(B), \tag{4.38}$$

and a matrix $X \geqslant 0$ satisfying (4.38) can be written as $X = -S^*M^{\dagger}S + U$, where $U > 0$.

(f) *The minimal negative signature of $p(X)$ subject to $X \geqslant 0$ is*

$$\min_{X \geqslant 0} i_{-}(A - BXB^*) = 0, \tag{4.39}$$

and a matrix $X \geqslant 0$ satisfying (4.39) is $X = -S^*M^{\dagger}S$.

In particular,

(g) *There exists an* $X \geqslant 0$ *such that* $A - BXB^* > 0$ *if and only if* $r(A) = m$.
(h) *There exists an* $X \geqslant 0$ *such that* $A - BXB^* < 0$ *if and only if* $r(B) = m$.
(i) *There always exists an* $X \geqslant 0$ *such that* $A - BXB^* \geqslant 0$, *say* $X = 0$.
(j) *There exists an* $X \geqslant 0$ *such that* $A - BXB^* \leqslant 0$ *if and only if* $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.

Proof. Under $A \geqslant 0$, substituting (1.5) and Lemma 2.3(d) into (4.13) and (4.14) yields

$$i_+(A - BXB^*) = r[A, B] - r(B) + i_-[F_{S_1}(X + S^*M^\dagger S)F_{S_1}], \tag{4.40}$$

$$i_-(A - BXB^*) = i_+[F_{S_1}(X + S^*M^\dagger S)F_{S_1}], \tag{4.41}$$

$$r(A - BXB^*) = r[A, B] - r(B) + r[F_{S_1}(X + S^*M^\dagger S)F_{S_1}]. \tag{4.42}$$

It can be derived from (1.9) and (3.3) that under $A \geqslant 0$,

$$r(S^*M^\dagger S) = i_-(S^*M^\dagger S) = r(A) + r(B) - r[A, B].$$

Hence $-S^*M^\dagger S$ is nonnegative definite from Lemma 1.3(d). In this event, we can derive from (1.5) and (4.32) that

$$\begin{aligned}\max_{X \geqslant 0} i_+[F_{S_1}(X + S^*M^\dagger S)F_{S_1}] &= \max_{X \geqslant 0} r[F_{S_1}(X + S^*M^\dagger S)F_{S_1}] \\ &= r(F_{S_1}) = r(M) - r[A, B] = r(B),\end{aligned} \tag{4.43}$$

and a matrix $X \geqslant 0$ satisfying this equality can be written, for example, as $X = -S^*M^\dagger S + U$, where $U > 0$. Substituting (4.43) into (4.41) and (4.42) gives (4.34) and (4.38).

Note that

$$\min_{X \geqslant 0} r[F_{S_1}(X + S^*M^\dagger S)F_{S_1}] = 0, \tag{4.44}$$

and a matrix $X \geqslant 0$ satisfying this equality can be written, for example, as $X = -S^*M^\dagger S \geqslant 0$. Substituting (4.44) into (4.42) gives (4.35).

Note that

$$\max_{X \geqslant 0} i_-[F_{S_1}(X + S^*M^\dagger S)F_{S_1}] = r[F_{S_1}(S^*M^\dagger S)F_{S_1}] = r(A) + r(B) - r[A, B], \tag{4.45}$$

and a matrix $X \geqslant 0$ satisfying this equality is $X = 0$. Substituting (4.45) into (4.40) gives (4.36).

Note that

$$\min_{X \geqslant 0} i_-[F_{S_1}(X + S^*M^\dagger S)F_{S_1}] = 0, \tag{4.46}$$

and any $X \geqslant -S^*M^\dagger S$ satisfies (4.46). Substituting (4.46) into (4.40) gives (4.37).

Finally note that

$$\min_{X \geqslant 0} i_+[F_{S_1}(X + S^*M^\dagger S)F_{S_1}] = 0, \tag{4.47}$$

and a matrix $X \geqslant 0$ satisfying this equality can be written as $X = -S^*M^\dagger S$. Substituting (4.47) into (4.41) gives (4.39). Results (g)–(j) follow from (4.36)–(4.39) and Lemma 1.4. $\square$

Remark 4.6. Set $Z = -BXB^*$. Then the matrix expression in (4.1) can be represented as a perturbed form of the Hermitian matrix A:

$$p(Z) = A + Z, \tag{4.48}$$

where Z is a Hermitian matrix satisfying $\mathcal{R}(Z) \subseteq \mathcal{R}(B)$. In this event, the results in Theorems 4.3 and 4.4 can be rewritten as some formulas for the rank and the positive and negative signatures of the Hermitian matrix A subject to a Hermitian perturbation $\mathcal{R}(Z) \subseteq \mathcal{R}(B)$. In particular, if $Z = \alpha uu^*$, where

u is a column vector and α is a real scalar, then (4.48) is called the rank one Hermitian perturbation of the Hermitian matrix A. Some formulas for the inertias of (4.48) with a rank one Hermitian perturbation were given in [17].

In addition, some problems on the positive and negative signatures of the matrix expression $A - BXB^*$ are worth for further consideration. For example,

(i) find Hermitian matrices Xs that satisfy the two equations $i_\pm(A - BXB^*) + i_\pm(BXB^*) = i_\pm(A)$, respectively;
(ii) derive the extremal values of the positive and negative signatures of $A - BXB^*$ with respect to a Hermitian matrix X under some restrictions, such as $CXC^* = D$;
(iii) find a Hermitian matrix X that satisfies the inequalities $A_1 \leqslant BXB^* \leqslant A_2$;
(iv) derive the extremal values of the positive and negative signatures for the Hermitian part of $A - BXC$. Some previous work on inertias of Hermitian part of a special case $A + BX$ can be found in [24].

5. Rank and inertia of the Hermitian solution to the matrix equation $AX = B$

Lemma 5.1 [26]. *Let $A, B \in \mathbb{C}^{m\times n}$ be given. Then,*

(a) *The matrix equation*

$$AX = B \tag{5.1}$$

has a Hermitian solution if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $AB^* = BA^*$. In this case, the general Hermitian solution to (5.1) can be written as

$$X = A^\dagger B + (A^\dagger B)^* - A^\dagger BA^\dagger A + F_AWF_A, \tag{5.2}$$

where $W \in \mathbb{C}_h^{n\times n}$ is arbitrary. In particular, (5.1) has a unique Hermitian solution if and only if $r(A) = n$.

(b) *The matrix equation in (5.1) has a solution $X \geqslant 0$ if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, $AB^* \geqslant 0$ and $r(AB^*) = r(B)$. In this case, the general solution to (5.1) can be written as*

$$X = B^*(AB^*)^\dagger B + F_AWF_A, \tag{5.3}$$

where $0 \leqslant W \in \mathbb{C}^{n\times n}$ is arbitrary. In particular, (5.1) has a unique solution $X \geqslant 0$ if and only if $r(A) = n$.

Note that both $A^\dagger B + B^*(A^\dagger)^*$ and F_A are Hermitian, while the matrix $A^\dagger AB^*(A^\dagger)^*$ is Hermitian as well under the assumption $AB^* = BA^*$. Hence (5.2) could be regarded as a special case of (4.1). Applying Theorem 4.4 to (5.2) produces the following result on the rank and the positive and negative signatures of Hermitian solution of the matrix equation $AX = B$.

Theorem 5.2. *Let $A, B \in \mathbb{C}^{m\times n}$ be given, and assume that the matrix equation $AX = B$ has a solution $X \in \mathbb{C}_h^{n\times n}$. Then*

$$\max_{AX=B,\, X\in\mathbb{C}_h^{n\times n}} r(X) = n + r(B) - r(A), \tag{5.4}$$

$$\min_{AX=B,\, X\in\mathbb{C}_h^{n\times n}} r(X) = 2r(B) - r(AB^*), \tag{5.5}$$

$$\max_{AX=B,\, X\in\mathbb{C}_h^{n\times n}} i_\pm(X) = n + i_\pm(AB^*) - r(A), \tag{5.6}$$

$$\min_{AX=B,\, X\in\mathbb{C}_h^{n\times n}} i_\pm(X) = r(B) - i_\mp(AB^*). \tag{5.7}$$

Hence,

(a) *$AX = B$ has a nonsingular Hermitian solution if and only if $r(A) = r(B)$.*
(b) *Any Hermitian solution to $AX = B$ is nonsingular if and only if $r(AB^*) = 2r(B) - n$.*
(c) *$AX = B$ has a solution $X > 0$ ($X < 0$) if and only if $AB^* \geqslant 0$ and $r(A) = r(B)$ ($AB^* \leqslant 0$ and $r(A) = r(B)$).*
(d) *Any solution to $AX = B$ satisfies $X > 0$ ($X < 0$) if and only if $AB^* \geqslant 0$ and $r(AB^*) = r(A)$ ($AB^* \leqslant 0$ and $r(AB^*) = r(A)$).*
(e) *$AX = B$ has a solution $X \geqslant 0$ ($X \leqslant 0$) if and only if $AB^* \geqslant 0$ and $r(AB^*) = r(B)$ ($AB^* \leqslant 0$ and $r(AB^*) = r(B)$).*
(f) *Any solution to $AX = B$ satisfies $X \geqslant 0$ ($X \leqslant 0$) if and only if $AB^* \geqslant 0$ and $r(A) = n$ ($AB^* \leqslant 0$ and $r(A) = n$).*
(g) *The following statements are equivalent:*

  (i) *The ranks of the Hermitian solutions of $AX = B$ are the same.*
  (ii) *The positive signatures of the Hermitian solutions of $AX = B$ are the same.*
  (iii) *The negative signatures of the Hermitian solutions of $AX = B$ are the same.*
  (iv) $r(AB^*) = r(A) + r(B) - n$.
  (v) $(I_n - A^\dagger A)(I_n - B^\dagger B) = 0$.
  (vi) $\mathcal{N}(A) \subseteq \mathcal{R}(B^*)$.

Proof. Applying (4.15)–(4.18) to (5.2) and simplifying by (1.2), (1.4), (2.5), $AA^\dagger B = B, AB^* = BA^*$ and Lemma 1.6(b), we obtain

$$\begin{aligned}
&\max_{AX=B, X\in\mathbb{C}_h^{n\times n}} r(X)\\
&= \max_{V\in\mathbb{C}_h^{n\times n}} r[A^\dagger B + B^*(A^\dagger)^* - A^\dagger AB^*(A^\dagger)^* + F_AVF_A]\\
&= r[A^\dagger B + B^*(A^\dagger)^* - A^\dagger AB^*(A^\dagger)^*, F_A]\\
&= r[A^\dagger B + A^\dagger AB^*(A^\dagger)^* - A^\dagger AB^*(A^\dagger)^*] + r(F_A) = n + r(B) - r(A),\\
&\max_{AX=B, X\in\mathbb{C}_h^{n\times n}} i_\pm(X)\\
&= \max_{V\in\mathbb{C}_h^{n\times n}} i_\pm[A^\dagger B + B^*(A^\dagger)^* - A^\dagger AB^*(A^\dagger)^* + F_AVF_A]\\
&= i_\pm\begin{bmatrix} A^\dagger B + B^*(A^\dagger)^* - A^\dagger AB^*(A^\dagger)^* & F_A \\ F_A & 0 \end{bmatrix}\\
&= i_\pm\{A[A^\dagger B + B^*(A^\dagger)^* - A^\dagger AB^*(A^\dagger)^*]A^*\} + r(F_A)\\
&= i_\pm[AB^*(A^\dagger)^*A^*] + r(F_A)\\
&= n + i_\pm(AB^*) - r(A),\\
&\min_{AX=B, X\in\mathbb{C}_h^{n\times n}} r(X)\\
&= \min_{V\in\mathbb{C}_h^{n\times n}} r[A^\dagger B + B^*(A^\dagger)^* - A^\dagger AB^*(A^\dagger)^* + F_AVF_A]\\
&= 2r[A^\dagger B + B^*(A^\dagger)^* - A^\dagger AB^*(A^\dagger)^*, F_A] - r\begin{bmatrix} A^\dagger B + B^*(A^\dagger)^* - A^\dagger AB^*(A^\dagger)^* & F_A \\ F_A & 0 \end{bmatrix}
\end{aligned}$$

$$= 2r(B) - r(AB^*),$$

$$\min_{AX=B, X\in\mathbb{C}_h^{n\times n}} i_{\pm}(X)$$

$$= \min_{V\in\mathbb{C}_h^{n\times n}} i_{\pm}[A^{\dagger}B + B^*(A^{\dagger})^* - A^{\dagger}AB^*(A^{\dagger})^* + F_AVF_A]$$

$$= r[A^{\dagger}B + B^*(A^{\dagger})^* - A^{\dagger}AB^*(A^{\dagger})^*, F_A] - i_{\mp}\begin{bmatrix} A^{\dagger}B + B^*(A^{\dagger})^* - A^{\dagger}AB^*(A^{\dagger})^* & F_A \\ F_A & 0 \end{bmatrix}$$

$$= r(B) - i_{\mp}(AB^*),$$

as required for (5.4)–(5.7). Results (a)–(f) follow from (5.4)–(5.7) and Lemma 1.4. The equivalence of (i)–(iv) in (g) follows from (5.4)–(5.7), and the equivalence of (iv)–(vi) in (g) follows from the formula $r(AB^*) = r(A) + r(B) - n + (I_n - A^{\dagger}A)(I_n - B^{\dagger}B)$; see [29, Theorem 6]. □

Furthermore, we have the following result.

Theorem 5.3. *Let $A, B \in \mathbb{C}^{m\times n}$ and $P \in \mathbb{C}_h^{n\times n}$ be given, and assume that the matrix equation $AX = B$ has a solution $X \in \mathbb{C}_h^{n\times n}$. Then*

$$\max_{AX=B, X\in\mathbb{C}_h^{n\times n}} i_{\pm}(P - X) = n + i_{\pm}(APA^* - AB^*) - r(A), \tag{5.8}$$

$$\min_{AX=B, X\in\mathbb{C}_h^{n\times n}} i_{\pm}(P - X) = r(AP - B) - i_{\mp}(APA^* - AB^*). \tag{5.9}$$

Hence,

(a) *$AX = B$ has a solution satisfying $P - X > 0$ ($P - X < 0$) if and only if $APA^* \geqslant AB^*$ and $r(APA^* - AB^*) = r(A)$ ($APA^* \leqslant AB^*$ and $r(APA^* - AB^*) = r(A)$).*
(b) *$AX = B$ has a solution satisfying $P - X \geqslant 0$ ($P - X \leqslant 0$) if and only if $APA^* \geqslant AB^*$ and $r(APA^* - AB^*) = r(AP - B)$ ($APA^* \leqslant AB^*$ and $r(APA^* - AB^*) = r(AP - B)$).*

Proof. Substituting (5.2) into $P - X$ gives

$$P - X = P - A^{\dagger}B - B^*(A^{\dagger})^* + A^{\dagger}AB^*(A^{\dagger})^* - F_AVF_A.$$

Applying (4.17) and (4.18) to it and simplifying by (1.2), (1.4), (2.5) and Lemma 1.6(b) gives

$$\max_{AX=B, X\in\mathbb{C}_h^{n\times n}} i_{\pm}(P - X)$$

$$= \max_{V\in\mathbb{C}_h^{n\times n}} i_{\pm}[P - A^{\dagger}B - B^*(A^{\dagger})^* + A^{\dagger}AB^*(A^{\dagger})^* - F_AVF_A]$$

$$= i_{\pm}\begin{bmatrix} P - A^{\dagger}B - B^*(A^{\dagger})^* + A^{\dagger}AB^*(A^{\dagger})^* & F_A \\ F_A & 0 \end{bmatrix}$$

$$= i_{\pm}[A^{\dagger}APA^{\dagger}A - A^{\dagger}AB^*(A^{\dagger})^*] + r(F_A)$$

$$= n + i_{\pm}(APA^* - AB^*) - r(A),$$

$$\min_{AX=B, X\in\mathbb{C}_h^{n\times n}} i_{\pm}(P - X)$$

$$= \min_{V\in\mathbb{C}_h^{n\times n}} i_{\pm}[P - A^{\dagger}B - B^*(A^{\dagger})^* + A^{\dagger}AB^*(A^{\dagger})^* - F_AVF_A]$$

$$= r[P - A^{\dagger}B - B^*(A^{\dagger})^* + A^{\dagger}AB^*(A^{\dagger})^*, F_A]$$
$$- i_{\mp}\begin{bmatrix} P - A^{\dagger}B - B^*(A^{\dagger})^* + A^{\dagger}AB^*(A^{\dagger})^* & F_A \\ F_A & 0 \end{bmatrix}$$
$$= r(AP - AA^{\dagger}B) - i_{\mp}[A^{\dagger}APA^{\dagger}A - A^{\dagger}AB^*(A^{\dagger})^*]$$
$$= r(AP - B) - i_{\mp}(APA^* - AB^*),$$

establishing (5.8) and (5.9). Results (a) and (b) follow from (5.8), (5.9) and Lemma 1.4. □

6. Extremal values of the rank and inertia of a partial Hermitian matrix

In this section, we consider the positive and negative signatures of the following 2×2 block Hermitian matrix

$$M(X) = \begin{bmatrix} A & B \\ B^* & X \end{bmatrix}, \tag{6.1}$$

where $A \in \mathbb{C}_h^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$ are given, and $X \in \mathbb{C}_h^{n\times n}$ is a variable matrix. Note that (6.1) can be represented as

$$M(X) = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_n \end{bmatrix} X[0, I_n], \tag{6.2}$$

that is, (6.1) is a special case of (4.1). Hence the extremal positive and negative signatures of (6.1) with respect to $X \in \mathbb{C}_h^{n\times n}$ can be derived from Theorem 4.4. On the other hand, applying (2.9) to (6.1) gives the following equalities

$$i_{\pm}[M(X)] = r[A, B] - i_{\mp}(A) + i_{\pm}[F_{B_1}(X - B^*A^{\dagger}B)F_{B_1}], \tag{6.3}$$

$$r[M(X)] = 2r[A, B] - r(A) + r[F_{B_1}(X - B^*A^{\dagger}B)F_{B_1}], \tag{6.4}$$

where $B_1 = E_AB$, and $F_{B_1}(X - B^*A^{\dagger}B)F_{B_1}$ is a homogeneous Hermitian matrix expression with respect to $Y = X - B^*A^{\dagger}B$.

Theorem 6.1. *Let $M(X)$ be as given in (6.1). Then*

$$\max_{X\in\mathbb{C}_h^{n\times n}} r[M(X)] = n + r[A, B], \tag{6.5}$$
$$\min_{X\in\mathbb{C}^{n\times n}} r[M(X)] = 2r[A, B] - r(A), \tag{6.6}$$
$$\max_{X\in\mathbb{C}_h^{n\times n}} i_{\pm}[M(X)] = n + i_{\pm}(A), \tag{6.7}$$
$$\min_{X\in\mathbb{C}_h^{n\times n}} i_{\pm}[M(X)] = r[A, B] - i_{\mp}(A). \tag{6.8}$$

(a) *The general expression of $X \in \mathbb{C}_h^{n\times n}$ satisfying (6.5) can be written as*

$$X = B^*A^{\dagger}B + U, \tag{6.9}$$

where $U \in \mathbb{C}_h^{n\times n}$ is chosen such that $i_+(F_{B_1}UF_{B_1}) = r(F_{B_1})$, say, $U = I_n$.

(b) *The general expression of $X \in \mathbb{C}_h^{n\times n}$ satisfying $i_+[M(X)] = i_+(A) + n$ can be written as*

$$X = B^*A^{\dagger}B + UU^*, \tag{6.10}$$

where $U \in \mathbb{C}_h^{n\times k}$ is chosen such that $i_+(F_{B_1}UU^*F_{B_1}) = r(F_{B_1})$, say, $U = I_n$; the general expression of X satisfying $i_-[M(X)] = i_-(A) + n$ can be written as

$$X = B^*A^\dagger B - UU^*, \tag{6.11}$$

where $U \in \mathbb{C}_h^{n\times k}$ is chosen such that $i_-(-F_{B_1}UU^*F_{B_1}) = r(F_{B_1})$, say, $U = I_n$.

(c) *The general expression of $X \in \mathbb{C}_h^{n\times n}$ satisfying (6.6) can be written as*

$$X = B^*A^\dagger B + B_1^*V^* + VB_1, \tag{6.12}$$

where $V \in \mathbb{C}^{n\times n}$ is arbitrary.

(d) *The general expression of $X \in \mathbb{C}_h^{n\times n}$ satisfying $i_+[M(X)] = r[A, B] - i_-(A)$ can be written as*

$$X = B^*A^\dagger B + UU^* + V - F_{B_1}VF_{B_1}, \tag{6.13}$$

where $U \in \mathbb{C}^{n\times k}$ and $V \in \mathbb{C}_h^{n\times n}$ are arbitrary.

(e) *The general expression of $X \in \mathbb{C}_h^{n\times n}$ satisfying $i_-[M(X)] = r[A, B] - i_+(A)$ can be written as*

$$X = B^*A^\dagger B - UU^* + V - F_{B_1}VF_{B_1}, \tag{6.14}$$

where $U \in \mathbb{C}^{n\times k}$ and $V \in \mathbb{C}_h^{n\times n}$ are arbitrary.

(f) *There exists an $X \in \mathbb{C}_h^{n\times n}$ such that $M(X)$ is nonsingular if and only if $r[A, B] = m$.*

(g) *$M(X)$ is nonsingular for all $X \in \mathbb{C}_h^{n\times n}$ if and only if $r(A) = m - n$, $r(B) = n$ and $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$.*

(h) *There exists an $X \in \mathbb{C}_h^{n\times n}$ such that $M(X) > 0$ $(M(X) < 0)$ if and only if $A > 0$ $(A < 0)$.*

(i) *There exists an $X \in \mathbb{C}_h^{n\times n}$ such that $M(X) \geqslant 0$ $(M(X) \leqslant 0)$ if and only if $r[A, B] = i_+(A)$, i.e., $A \geqslant 0$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ $(r[A, B] = i_-(A)$, i.e., $A \leqslant 0$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A))$.*

(j) *$r[M(X)]$ is invariant with respect to $X \in \mathbb{C}_h^{n\times n}$ $\Leftrightarrow$ $i_+[M(X)]$ is invariant with respect to $X \in \mathbb{C}_h^{n\times n}$ $\Leftrightarrow$ $i_-[M(X)]$ is invariant with respect to $X \in \mathbb{C}_h^{n\times n}$ $\Leftrightarrow$ $r(B) = n$ and $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$.*

Proof. It follows from (6.3) and (6.4) that

$$\max_{X\in\mathbb{C}_h^{n\times n}} r[M(X)] = 2r[A, B] - r(A) + \max_{Y\in\mathbb{C}_h^{n\times n}} r(F_{B_1}YF_{B_1}), \tag{6.15}$$

$$\min_{X\in\mathbb{C}_h^{n\times n}} r[M(X)] = 2r[A, B] - r(A) + \min_{Y\in\mathbb{C}_h^{n\times n}} r(F_{B_1}YF_{B_1}), \tag{6.16}$$

$$\max_{X\in\mathbb{C}_h^{n\times n}} i_\pm[M(X)] = r[A, B] - i_\mp(A) + \max_{Y\in\mathbb{C}_h^{n\times n}} i_\pm(F_{B_1}YF_{B_1}), \tag{6.17}$$

$$\min_{X\in\mathbb{C}_h^{n\times n}} i_\pm[M(X)] = r[A, B] - i_\mp(A) + \min_{Y\in\mathbb{C}_h^{n\times n}} i_\pm(F_{B_1}YF_{B_1}), \tag{6.18}$$

where $Y = X - B^*A^\dagger B$. It can be derived from (1.2) that

$$r(F_{B_1}) = n - r(E_AB) = n + r(A) - r[A, B].$$

It is also obvious that

$$\max_{Y\in\mathbb{C}_h^{n\times n}} r(F_{B_1}YF_{B_1}) = \max_{Y\in\mathbb{C}_h^{n\times n}} i_\pm(F_{B_1}YF_{B_1}) = r(F_{B_1}) = n + r(A) - r[A, B],$$

$$\min_{Y\in\mathbb{C}_h^{n\times n}} r(F_{B_1}YF_{B_1}) = \min_{Y\in\mathbb{C}_h^{n\times n}} i_\pm(F_{B_1}YF_{B_1}) = 0,$$

and the matrices Xs satisfying these equalities can be written as (6.9)–(6.14), respectively. Substituting them into (6.15)–(6.18) yields (6.5)–(6.8). Results (f)–(j) follow from (6.5)–(6.8) and Lemma 1.4. $\square$

7. The extremal inertias of $A - B_1X_1B_1^* - \cdots - B_kX_kB_k^*$

Without much effort, the results in Section 4 can be extended to the following matrix expression

$$p(X_1, \ldots, X_k) = A - B_1X_1B_1^* - \cdots - B_kX_kB_k^*, \tag{7.1}$$

where $A \in \mathbb{C}_h^{m\times m}$ and $B_i \in \mathbb{C}^{m\times n_i}$ are given, and $X_i \in \mathbb{C}_h^{n_i\times n_i}$, $i = 1, \ldots, k$, are variable matrices.

Theorem 7.1. *Let $p(X_1, \ldots, X_k)$ be as given in (7.1), and denote $B = [B_1, \ldots, B_k]$ and $M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}$. Then*

$$\max_{X_i\in\mathbb{C}_h^{n_i\times n_i}, i=1,\ldots,k} i_\pm[p(X_1, \ldots, X_k)] = i_\pm(M), \tag{7.2}$$

$$\min_{X_i\in\mathbb{C}_h^{n_i\times n_i}, i=1,\ldots,k} i_\pm[p(X_1, \ldots, X_k)] = r[A, B] - i_\mp(M). \tag{7.3}$$

Hence,

(a) *There exist $X_i \in \mathbb{C}_h^{n_i\times n_i}$, $i = 1, \ldots, k$, such that*

$$p(X_1, \ldots, X_k) > 0 \quad (< 0) \tag{7.4}$$

if and only if $i_+(M) = m$, or equivalently $E_BAE_B \geqslant 0$ and $r(E_BAE_B) = r(E_B)$ ($i_-(M) = m$, or equivalently $E_BAE_B \leqslant 0$ and $r(E_BAE_B) = r(E_B)$).

(b) *There exist $X_i \in \mathbb{C}_h^{n_i\times n_i}$, $i = 1, \ldots, k$, such that*

$$p(X_1, \ldots, X_k) \geqslant 0 \quad (\leqslant 0) \tag{7.5}$$

if and only if $i_+(M) = r[A, B]$ ($i_-(M) = r[A, B]$).

Proof. Eq. (7.1) can be written as

$$p(X_1, \ldots, X_k) = A - BXB^*, \quad X = \mathrm{diag}(X_1, \ldots, X_k). \tag{7.6}$$

Applying (4.13) to it gives

$$i_\pm[p(X_1, \ldots, X_k)] = r[A, B] - i_\mp(M) + i_\mp[F_{S_1}(X + S^*M^\dagger S)F_{S_1}], \tag{7.7}$$

where $S = \begin{bmatrix} 0 \\ I_{n_1+\cdots+n_k} \end{bmatrix}$ and $S_1 = S - MM^\dagger S$. Set $X = tI_{n_1+\cdots+n_k}$ for sufficiently large $t > 0$. Then $X + S^*M^\dagger S$ is positive definite, and

$$i_+[F_{S_1}(X + S^*M^\dagger S)F_{S_1}] = r(F_{S_1}) = r(M) - r[A, B],$$

$$i_-[F_{S_1}(X + S^*M^\dagger S)F_{S_1}] = 0.$$

Also set $X = tI_{n_1+\cdots+n_k}$ for sufficiently small $t < 0$. Then $X + S^*M^\dagger S$ is negative definite, and

$$i_+[F_{S_1}(X + S^*M^\dagger S)F_{S_1}] = 0,$$
$$i_-[F_{S_1}(X + S^*M^\dagger S)F_{S_1}] = r(F_{S_1}) = r(M) - r[A, B].$$

Substituting them into (7.7) gives

$$\begin{aligned}\max_X i_\pm[p(X_1, \ldots, X_k)] &= r[A, B] - i_\mp(M) + \max_X i_\mp[F_{S_1}(X + S^*M^\dagger S)F_{S_1}] \\ &= r[A, B] - i_\mp(M) + r(M) - r[A, B] \\ &= i_\pm(M),\end{aligned}$$

$$\begin{aligned}\min_X i_{\pm}[p(X_1, \ldots, X_k)] &= r[A, B] - i_{\mp}(M) + \min_X i_{\mp}[F_{S_1}\left(Z + S^*M^{\dagger}S\right)F_{S_1}] \\ &= r[A, B] - i_{\mp}(M),\end{aligned}$$

establishing (7.2) and (7.3). Results (a) and (b) follow from (7.2), (7.3) and Lemma 1.4. □

Corollary 7.2. *Let $p(X_1, \ldots, X_k)$ be as given in (7.1), and denote $B = [B_1, \ldots, B_k]$. Also assume $A \geqslant 0$. Then*

$$\max_{X_i \in \mathbb{C}_h^{n_i \times n_i}, i=1,\ldots,k} i_+(A - B_1X_1B_1^* - \cdots - B_kX_kB_k^*) = r[A, B], \tag{7.8}$$

$$\min_{X_i \in \mathbb{C}_h^{n_i \times n_i}, i=1,\ldots,k} i_+(A - B_1X_1B_1^* - \cdots - B_kX_kB_k^*) = r[A, B] - r(B), \tag{7.9}$$

$$\max_{X_i \in \mathbb{C}_h^{n_i \times n_i}, i=1,\ldots,k} i_-(A - B_1X_1B_1^* - \cdots - B_kX_kB_k^*) = r(B), \tag{7.10}$$

$$\min_{X_i \in \mathbb{C}_h^{n_i \times n_i}, i=1,\ldots,k} i_-(A - B_1X_1B_1^* - \cdots - B_kX_kB_k^*) = 0. \tag{7.11}$$

8. Concluding remarks

In the previous sections, we gave a variety of formulas for (extremal) positive and negative signatures of Hermitian matrices through generalized inverses of matrices and some elementary congruent matrix operations, and presented many consequences and applications of these formulas. Most of results obtained are simple and elementary, so that they are easy to use in the investigations of Hermitian matrices and their applications.

Motivated by the work in this paper, more problems on inertias of Hermitian matrices can be proposed and studied. For example, it would be of interest to derive the extremal inertias of $A \pm BXB^*$ with respect to a Hermitian solution of a consistent matrix equation $CXC^* = D$. Furthermore, it is worthwhile to consider the extremal ranks and inertias of some general matrix expressions with respect to variable matrices, such as,

$$\begin{aligned}&A - BXB^* \pm CXC^*, \quad X = \pm X^*, \\ &A - BX \pm (BX)^*, \quad A - BXC \pm (BXC)^*, \quad E - (A - BXC)D(A - BXC)^*.\end{aligned}$$

In an earlier paper [25], Johnson and Lundquist defined the inertia of a Hermitian operator on a Hilbert space, and gave some formulas for inertias of Hermitian operators and their inverses. Under this general frame, it would be of interest to extend the results in this paper to inertias of Hermitian operators on a Hilbert space.

Acknowledgments

The author thanks an anonymous referee and the handling editor for their helpful suggestions and constructive comments on earlier versions of this paper.

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